

Spreading Models in Banach Space Theory

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ABSTRACT. We extend the classical Brunel-Sucheston definition of the spreading model by introducing the \mathcal{F} -sequences $(x_s)_{s \in \mathcal{F}}$ in a Banach space and the plegma families in \mathcal{F} where \mathcal{F} is a regular thin family. The new concept yields a transfinite increasing hierarchy of classes of 1-subsymmetric sequences. We explore the corresponding theory and we present examples establishing this hierarchy and illustrating the limitation of the theory.

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⁴This work is part of the PhD Thesis of the third named author

Introduction

Spreading models have been invented by A. Brunel and L. Sucheston in the middle of 70's (c.f. [5]) and since then they have a constant presence in the evolution of the Banach space theory. Since the goal of the present monograph is to extend and study that notion, we begin by recalling the basics of their definition and some of their consequences.

The Brunel-Sucheston spreading models

A spreading model of a Banach space X is an 1-subsymmetric sequence¹ $(e_n)_n$ in a seminormed space $(E, \|\cdot\|_*)$ connected to X with an *asymptotic isometry* which we next describe. A sequence $(x_n)_n$ in a Banach space $(X, \|\cdot\|)$ is *asymptotically isometric* to a sequence $(e_n)_n$, as above, if there exists a null sequence $(\delta_n)_n$ of positive reals such that for every $F = \{n_1 < \dots < n_k\}$ with $n_1 \leq |F|$ and every $(a_i)_{i=1}^k \in [-1, 1]^k$, we have that

$$\left| \left\| \sum_{i=1}^k a_i x_{n_i} \right\| - \left\| \sum_{i=1}^k a_i e_i \right\|_* \right| < \delta_{n_1}$$

A sequence $(e_n)_n$ is a spreading model of the space X if there exists a sequence $(x_n)_n$ in X which is asymptotically isometric to $(e_n)_n$. In this case we say that $(x_n)_n$ generates $(e_n)_n$ as a spreading model. By applying Ramsey theorem (c.f. [26]), Brunel and Sucheston proved that every bounded sequence $(x_n)_n$ in a Banach space X has a subsequence $(x_{k_n})_n$ generating some $(e_n)_n$ as a spreading model.

The 1-subsymmetric sequences possess regular structure. For instance if $(e_n)_n$ is weakly null, then it is 1-unconditional. Furthermore if $(e_n)_n$ is unconditional then either it is equivalent to the standard basis of ℓ^1 or it is norm Cesàro summable to zero. The importance of the spreading models arises from the fact that they connect the structure of an arbitrary Banach space X to the corresponding one of spaces with 1-subsymmetric bases. For example every Banach space admits unconditional sequences $(e_n)_n$ as a spreading model and moreover every seminormalized weakly null sequence $(x_n)_n$ in a Banach space X contains a subsequence which either is norm Cesàro summable to zero or generates ℓ^1 as a spreading model. We should also add that recent discoveries (c.f. [13], [14]) have shown that similar regular structure is not expected inside a generic Banach space.

It is clear from the definition of the spreading model that it describes a kind of almost isometric finite representability² of the space generated by the sequence $(e_n)_n$ into the space $(X, \|\cdot\|)$. However there exists a significant difference between

¹A sequence $(e_n)_n$ in a seminormed space $(E, \|\cdot\|_*)$ is called 1-subsymmetric if for every $n \in \mathbb{N}$, $k_1 < \dots < k_n$ in \mathbb{N} and $a_1, \dots, a_n \in \mathbb{R}$ we have that $\|\sum_{j=1}^n a_j e_j\|_* = \|\sum_{j=1}^n a_j e_{k_j}\|_*$

²A Banach space Y is finitely representable in X if for every finite dimensional subspace F of Y and every $\varepsilon > 0$ there exists $T : F \rightarrow Y$ bounded linear injection such that $\|T\| \cdot \|T^{-1}\| < 1 + \varepsilon$.

the two concepts. Indeed in the frame of the finite representability there are two classical achievements: Dvoretzky's theorem (c.f. [6]) asserting that ℓ^2 is almost isometrically finitely representable in every Banach space X and also Krivine's theorem (c.f. [19]) asserting that for every linearly independent sequence $(x_n)_n$ in X there exists a $1 \leq p \leq \infty$ such that ℓ^p is almost isometrically block finitely representable in X . On the other hand there exists a reflexive space X admitting no ℓ^p as a spreading model (c.f. [24]). Thus the spreading models of a space lie strictly between the finitely representable spaces in X and the spaces that are isomorphic to a subspace of X .

We think of the spreading models associated to a Banach space X as a cloud of Banach spaces, including many members with regular structure, surrounding the space X and offering information concerning the local structure of X in an asymptotic manner. Our aim is to enlarge that cloud and to fill in the gap between spreading models and the spaces which are finitely representable in X . More precisely we extend the Brunel-Sucheston definition and we show that under the new definition the spreading models associated to a Banach space X form a whole hierarchy of classes of spaces indexed by the countable ordinals. In this hierarchy the classical spreading models correspond to the first class.

The extended definition of the spreading models

The extended notion heavily depends on the following two ingredients. The first one is the \mathcal{F} -sequences with \mathcal{F} a *regular thin* family of finite subsets of \mathbb{N} . The \mathcal{F} -sequences will replace the usual sequences. The second is the new concept of *plegma* families of subsets of \mathbb{N} . This is very crucial for our approach as it permits to apply Ramsey theory. Let us also note that the plegma families are invisible in the classical definition of spreading models. Next we shall describe in detail the aforementioned concepts and the definition of the spreading models. In the sequel for an infinite subset M of \mathbb{N} by $[M]^{<\infty}$ (resp. $[M]^\infty$) we denote the set of all finite (resp. infinite) subsets of M .

Thin families find their origin in [22] and were extensively studied in [25]. In the present work we will consider a special class of thin families defined as *regular thin* families (see Definition 1.1). An important feature of a thin family \mathcal{F} is the order of \mathcal{F} denoted as $o(\mathcal{F})$ and which is defined to be the height of the tree $\widehat{\mathcal{F}} = \{t \in [\mathbb{N}]^{<\infty} : \exists s \in \mathcal{F} \text{ with } t \sqsubseteq s\}$ associated to the family \mathcal{F} (see [18], [25]). Typical examples of regular thin families are the families of k -subsets of \mathbb{N} , $\mathcal{F}_k = [\mathbb{N}]^k$ with $o(\mathcal{F}_k) = k$, the maximal elements of the Schreier family, $\mathcal{F}_\omega = \{s \subset \mathbb{N} : \min s = |s|\}$ with $o(\mathcal{F}_\omega) = \omega$ and a generic one is the family \mathcal{F}_{ω^ξ} of the maximal elements of the ξ -Schreier family \mathcal{S}_ξ , with $o(\mathcal{F}_{\omega^\xi}) = \omega^\xi$. We shall consider \mathcal{F} -sequences $(x_s)_{s \in \mathcal{F}}$, in some set, as well as \mathcal{F} -subsequences $(x_s)_{s \in \mathcal{F} \upharpoonright L}$, where $L \in [\mathbb{N}]^\infty$ and $\mathcal{F} \upharpoonright L = \{s \in \mathcal{F} : s \subset L\}$.

The plegma families are some special finite sequences of subsets of \mathbb{N} (see Definition 1.7). Roughly speaking they are pairwise disjoint families $\{s_1, \dots, s_l\}$ of finite subsets of \mathbb{N} satisfying the following property. The first elements of $\{s_i\}_{i=1}^l$ are in increasing order and they lie before their second elements which are also in increasing order and so on. The plegma families do not necessarily include sets of equal size. For each $l \in \mathbb{N}$, let $Plm_l(\mathcal{F})$ be the set of all plegma families $\{s_1, \dots, s_l\}$ with each $s_i \in \mathcal{F}$. The plegma families satisfy the following Ramsey property which is fundamental for this work.

Proposition 0.1. Let M be an infinite subset of \mathbb{N} , $l \in \mathbb{N}$ and \mathcal{F} be a regular thin family. Then for every finite coloring of $Plm_l(\mathcal{F} \upharpoonright M)$ there exists $L \in [M]^\infty$ such that $Plm_l(\mathcal{F} \upharpoonright L)$ is monochromatic.

As in the case of the classical spreading models, an iterated use of the above proposition, yields that for every bounded \mathcal{F} -sequence $(x_s)_{s \in \mathcal{F}}$ in a Banach space X there exist an infinite subset M of \mathbb{N} and a seminorm $\|\cdot\|_*$ on $c_{00}(\mathbb{N})$ under which the natural Hamel basis $(e_n)_n$ is an 1-subsymmetric sequence such that the following is satisfied: For every $l \in \mathbb{N}$, $a_1, \dots, a_l \in \mathbb{R}$ and every sequence $((s_i^n)_{i=1}^l)_n$ in $Plm_l(\mathcal{F} \upharpoonright M)$ with $\min s_1^n \rightarrow \infty$, we have

$$\left\| \sum_{i=1}^l a_i e_i \right\|_* = \lim_{n \rightarrow \infty} \left\| \sum_{i=1}^l a_i x_{s_i^n} \right\|$$

The sequence $(e_n)_n$ will be called an \mathcal{F} -spreading model of X which is generated by the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$.

Let us point out that the \mathcal{F} -sequences of order 1 coincide with the usual sequences and also the corresponding plegma families are the subsets of \mathbb{N} . Thus the above definition when $o(\mathcal{F}) = 1$ recovers the classical Brunel-Sucheston spreading models.

There are evidences supporting that the above definition is the appropriate extension of the classical one. The first one is that an \mathcal{F} -spreading model $(e_n)_n$ depends *only* on the order of \mathcal{F} . More precisely the following holds.

Proposition 0.2. Let X be a Banach space and \mathcal{F}, \mathcal{G} be regular thin families. If $o(\mathcal{F}) = o(\mathcal{G})$ then $(e_n)_n$ is an \mathcal{F} -spreading model of X if and only if $(e_n)_n$ is a \mathcal{G} -spreading model of X . More generally, if $o(\mathcal{F}) \leq o(\mathcal{G})$ and $(e_n)_n$ is an \mathcal{F} -spreading model of X then $(e_n)_n$ is a \mathcal{G} -spreading model of X .

The above allow us to classify the spreading models as a transfinite hierarchy as follows.

Definition 0.3. Let X be a Banach space and $1 \leq \xi < \omega_1$ be a countable ordinal. We will say that $(e_n)_n$ is a ξ -spreading model of X if there exists a regular thin family \mathcal{F} with $o(\mathcal{F}) = \xi$ such that $(e_n)_n$ is an \mathcal{F} -spreading model of X . The set of all ξ -spreading models of X will be denoted by $\mathcal{SM}_\xi(X)$.

Moreover the preceding proposition yields that the above defined transfinite hierarchy of spreading models is an increasing one i.e. for every Banach space X and $1 \leq \zeta < \xi < \omega_1$ we have that $\mathcal{SM}_\zeta(X) \subseteq \mathcal{SM}_\xi(X)$. An open problem here is whether this hierarchy is stabilized, i.e. if for every separable Banach space X there exists a countable ordinal ξ such that for every $\zeta > \xi$, $\mathcal{SM}_\zeta(X) = \mathcal{SM}_\xi(X)$.

Let us point out that the ξ -spreading models of X have a weaker asymptotic relation to the space X as ξ increases to ω_1 . A natural question arising from the above discussion is whether the ξ -spreading models of X , $\xi < \omega_1$, could recapture Krivine's theorem. As we will see this is not always true.

Other approaches

There are two other concepts in the literature sharing similar features with the aforementioned extended definition. The first appears in [15] and concerns the so called asymptotic models which are associated to $[\mathbb{N}]^2$ -bounded sequences in a Banach space. The asymptotic models are not necessarily subsymmetric sequences.

The second one is explicitly stated in [24] although was known to the experts of the Banach space theory. This concerns what we call *strong k -order* spreading models, which are inductively defined as follows. First we need some notation from [24]. Let X, E be Banach spaces. We write $X \rightarrow E$ if E has a Schauder basis which is a spreading model of some seminormalized basic sequence in X and $X \xrightarrow{k} E$ if $X \rightarrow E_1 \rightarrow \dots \rightarrow E_{k-1} \rightarrow E$ for some finite sequence E_1, \dots, E_{k-1} . Note that for every $k \in \mathbb{N}$ if $X \xrightarrow{k} E$ then E has a subsymmetric Schauder basis. An 1-subsymmetric Schauder basic sequence $(e_n)_n$ is said to be a *strong k -order* spreading model of a Banach space X if setting $E = \overline{\langle (e_n)_n \rangle}$ then $X \xrightarrow{k} E$. If in each inductive step we consider block sequences instead of Schauder basic ones, we define in a similar manner the *block strong k -order* spreading models $(X \xrightarrow[k]{bl} E)$. It is easy to see that the strong k -order spreading models define a countable hierarchy and a problem posed in [24] is whether there exists a Banach space X such that no strong k -order spreading model contains some ℓ^p $1 \leq p < \infty$ or c_0 .

It is interesting that the extended definition includes the strong k -order spreading models as follows.

Proposition 0.4. Let X be a Banach space and $k \in \mathbb{N}$. Then every block strong k -order spreading model is also a k -order spreading model. Moreover if all strong l -order spreading models are reflexive, for every $1 \leq l < k$, then all strong k -order spreading models are k -order ones.

This proposition enable us to answer the aforementioned problem by showing the following more general. There exists a reflexive Banach space X such that no ℓ^p , $1 \leq p < \infty$, or c_0 is embedded into $E = \overline{\langle (e_n)_n \rangle}$, where $(e_n)_n$ is a spreading model of any order of X . We also provide an example showing that the strong k -order spreading models do not coincide with the k -order ones.

An outline of the main results

We pass to present our results related to the extended spreading model notion. As we have already mentioned the dominant concepts for our approach are both plegma and regular thin families while the tools are mainly of combinatorial nature with the Ramsey theory keeping a central role.

We start with some results which explain the behavior of the plegma families. For a family \mathcal{F} of finite subsets of \mathbb{N} , by $Plm(\mathcal{F})$ we denote the set of all plegma families in \mathcal{F} , i.e. $Plm(\mathcal{F}) = \cup_{l=1}^{\infty} Plm_l(\mathcal{F})$.

Theorem 0.5. Let \mathcal{F}, \mathcal{G} be regular thin families. If $o(\mathcal{F}) \leq o(\mathcal{G})$ then for every $M \in [\mathbb{N}]^{\infty}$ there exist $N \in [\mathbb{N}]^{\infty}$ and a map $\varphi : \mathcal{G} \upharpoonright N \rightarrow \mathcal{F} \upharpoonright M$ such that for every $(s_i)_{i=1}^l \in Plm(\mathcal{F})$, we have that $(\varphi(s_i))_{i=1}^l \in Plm(\mathcal{G})$.

We could say that the above result is the set theoretic analogue of Proposition 0.2 above and it is the basic ingredient of its proof. In the other direction we prove the following which explicitly forbids the existence of such maps from lower to higher order regular thin families.

Theorem 0.6. Let \mathcal{F}, \mathcal{G} be regular thin families. If $o(\mathcal{F}) < o(\mathcal{G})$ then for every $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ and $M \in [\mathbb{N}]^{\infty}$ there exists $L \in [M]^{\infty}$ such that for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L$ neither $(\phi(s_1), \phi(s_2))$ nor $(\phi(s_2), \phi(s_1))$ is a plegma pair.

The proof of Theorem 0.6 relies on the *plegma paths* which are finite sequences $(s_i)_{i=1}^d$ such that (s_i, s_{i+1}) is a plegma pair. Let us point out that the relation

“(s_1, s_2) is a plegma pair” is neither symmetric nor transitive. In the context of graph theory the above theorem describes the following phenomenon. Viewing \mathcal{F} and \mathcal{G} as directed graphs with edges the plegma pairs in \mathcal{F} and \mathcal{G} respectively, we have that for every $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ and $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that φ embeds the associated to $\mathcal{F} \upharpoonright L$ graph into the complement of the corresponding one of \mathcal{G} .

We proceed to the study of the topological behavior of the \mathcal{F} -sequences. To this end we introduce the following definition.

Definition 0.7. Let (X, \mathcal{T}) be a topological space, \mathcal{F} be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in X . We say that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is *subordinated* (with respect to (X, \mathcal{T})) if there exists a continuous map $\widehat{\varphi} : \widehat{\mathcal{F}} \upharpoonright M \rightarrow (X, \mathcal{T})$ with $\widehat{\varphi}(s) = x_s$, for all $s \in \mathcal{F} \upharpoonright M$.

Let us say that an \mathcal{F} -sequence $(x_s)_{s \in \mathcal{F}}$ in a Banach space X is weakly relatively compact if the set $\overline{\{x_s : s \in \mathcal{F}\}}^w$ is weakly compact. The following proposition actually generalizes to weakly relatively compact \mathcal{F} -sequences the well known property that every sequence in a weakly relatively compact subset of a Banach space has a convergent subsequence.

Proposition 0.8. Let X be a Banach space, \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ be a weakly relatively compact \mathcal{F} -sequence in X . Then for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is subordinated.

Our next goal is to give sufficient conditions for \mathcal{F} -sequences yielding additional information concerning the structure of the generated spreading models. First we present analogues of the Brunel-Sucheston condition (c.f. [5]), which ensure that the seminorm in the space generated by the spreading model is actually a norm (see Theorem 3.16). Next we provide conditions for a spreading model being a Schauder basic sequence. In particular we prove the following.

Theorem 0.9. Let X be a Banach space, \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in X . Let $((e_n)_n, \|\cdot\|_*)$ be an \mathcal{F} -spreading model generated by $(x_s)_{s \in \mathcal{F}}$ such that $\|\cdot\|_*$ is a norm. If $\{x_s : s \in \mathcal{F}\}$ admits a Skipped Schauder Decomposition (SSD) then $(e_n)_n$ is Schauder basic.

(For the definition of (SSD) see Definition 3.27). The proof of the above theorem relies on the following combinatorial result.

Proposition 0.10. Let \mathcal{F} be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $\varphi : \mathcal{F} \rightarrow \mathbb{N}$. Then there exists $N \in [M]^\infty$ such that either φ is constant on $\mathcal{F} \upharpoonright N$ or for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright N$, $\varphi(s_2) - \varphi(s_1) > 1$.

Concerning the unconditional spreading models we have the following, which generalizes a well known fact for classical spreading models, namely that seminormalized weakly null sequences generate unconditional spreading models.

Theorem 0.11. Let \mathcal{F} be a regular thin family and $L \in [\mathbb{N}]^\infty$. Let $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ be a seminormalized \mathcal{F} -sequence in a Banach space X generating a spreading model $(e_n)_n$. Suppose that $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is subordinated with respect to the weak topology on X and let $\widehat{\varphi} : \widehat{\mathcal{F}} \upharpoonright L \rightarrow (X, w)$ be the continuous map witnessing it. If $\widehat{\varphi}(\emptyset) = 0$, then the sequence $(e_n)_n$ is unconditional.

We next present some results concerning spreading models which are generated by \mathcal{F} -sequences in a Banach space with a Schauder basis. We will need the following definition.

Definition 0.12. Let X be a Banach space with a Schauder basis. Let \mathcal{F} be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in X consisting of finitely supported vectors. We will say that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is *plegma disjointly supported* (resp. *plegma block*) if for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright M$ we have that $\text{supp}(x_{s_1}) \cap \text{supp}(x_{s_2}) = \emptyset$ (resp. $\text{supp}(x_{s_1}) < \text{supp}(x_{s_2})$).

The above definition replaces the well known concept of disjointly supported (resp. block) sequences which occur when we deal with sequences in Banach spaces with a Schauder basis. It is worth pointing out that when $o(\mathcal{F}) > 1$ we could not expect that the whole \mathcal{F} -sequence (or even an \mathcal{F} -subsequence) is either disjointly supported or block.

Theorem 0.13. Let X be a Banach space with a Schauder basis and \mathcal{F} be a regular thin family. Let $(e_n)_n$ be an \mathcal{F} -spreading model of X generated by a weakly relatively compact \mathcal{F} -sequence. If $(e_n)_n$ is Schauder basic then it is unconditional and it is also generated by a plegma disjointly supported \mathcal{F} -subsequence.

As we will see the sequence $(e_n)_n$ in the above theorem is not in general generated by a plegma block \mathcal{F} -subsequence. However when $(e_n)_n$ is equivalent to the standard basis of c_0 or ℓ_1 we have the following results.

Theorem 0.14. Let X be a Banach space with a Schauder basis. If X admits c_0 as a spreading model generated by a weakly relatively compact \mathcal{F} -sequence then X also admits c_0 as a plegma block generated spreading model.

The proof uses a combinatorial result concerning partial unconditionality of tree basic sequences. As consequence of the above we have the following.

Corollary 0.15. Let X be a reflexive Banach space with a Schauder basis. If X admits c_0 as a spreading model generated by a weakly relatively compact \mathcal{F} -sequence then X^* admits ℓ^1 as a plegma block spreading model.

It is well known that the above duality result does not hold in the inverse direction. Namely there are reflexive spaces admitting ℓ^1 as a classical spreading model and their dual does not admit c_0 . We also present a similar example for higher order spreading models.

An analogue of Theorem 0.14 for ℓ^1 spreading models also holds under an additional assumption. A Banach space X with a Schauder basis has the property \mathcal{P} if for every $\delta > 0$ there exists a $k \in \mathbb{N}$ such that for every finite block sequence $(x_i)_{i=1}^k$ in X with $\|x_i\| \geq \delta$ for all $1 \leq i \leq k$ we have that $\|\sum_{i=1}^k x_i\| > 1$.

Theorem 0.16. Let X be a Banach space with a Schauder basis and the property \mathcal{P} . Let also $(x_s)_{s \in \mathcal{F}}$ be a weakly relatively compact \mathcal{F} -sequence admitting ℓ^1 as an \mathcal{F} -spreading model. Then X admits ℓ^1 as a plegma block generated spreading model.

The above theorem is a key ingredient for showing that there exists a reflexive space admitting no ℓ_p as a spreading model. We will see that the additional assumption concerning property \mathcal{P} is a necessary one for the conclusion of the above theorem.

The next result concerns Cesàro summability for k -order \mathcal{F} -sequences, $k \in \mathbb{N}$. We first define the k -Cesàro summability in the following manner.

Definition 0.17. Let X be a Banach space, $x_0 \in X$, $k \in \mathbb{N}$, $(x_s)_{s \in [\mathbb{N}]^k}$ be a $[\mathbb{N}]^k$ -sequence in X and $M \in [\mathbb{N}]^\infty$. We will say that the $[\mathbb{N}]^k$ -subsequence $(x_s)_{s \in [M]^k}$ is

k -Cesàro summable to x_0 if

$$\binom{n}{k}^{-1} \sum_{s \in [M|n]^k} x_s \xrightarrow[n \rightarrow \infty]{\|\cdot\|} x_0$$

where $M|n = \{M(1), \dots, M(n)\}$.

We prove the following extension of a well known result of H. P. Rosenthal which corresponds to the case $k = 1$.

Theorem 0.18. Let X be a Banach space, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a weakly relatively compact $[\mathbb{N}]^k$ -sequence in X . Then there exists $M \in [\mathbb{N}]^\infty$ such that at least one of the following holds:

- (i) The subsequence $(x_s)_{s \in [M]^k}$ generates an $[\mathbb{N}]^k$ -spreading model equivalent to the standard basis of ℓ^1 .
- (ii) There exists $x_0 \in X$ such that for every $L \in [M]^\infty$ the subsequence $(x_s)_{s \in [L]^k}$ is k -Cesàro summable to x_0 .

There are significant differences between the cases $k = 1$ and $k \geq 2$. Firstly for $k = 1$ the two alternatives are exclusive but this does not remain valid for $k \geq 2$. Secondly the proof for the case $k \geq 2$ uses the multidimensional Szemerédi's theorem (c.f. [8]).

We also give some composition properties of spreading models. More precisely we have the following result.

Theorem 0.19. Let X be a Banach space and $(e_n)_n$ be a Schauder basic sequence in $\mathcal{SM}_\xi(X)$, for some $\xi < \omega_1$. Let $E = \overline{\langle (e_n)_n \rangle}$ and for some $k \in \mathbb{N}$, let $(\bar{e}_n)_n \in \mathcal{SM}_k(E)$ be a plegma block generated spreading model. Then

$$(\bar{e}_n)_n \in \mathcal{SM}_{\xi+k}(X)$$

The above yields the following concerning ℓ^p spreading models.

Proposition 0.20. Let X be a Banach space and $(e_n)_n$ be a ξ -order spreading model of X . Let also $E = \overline{\langle (e_n)_n \rangle}$. If for some $1 < p < \infty$ the space E contains an isomorphic copy of ℓ^p then X admits a $(\xi + 1)$ -order spreading model equivalent to the usual basis of ℓ^p .

Using additionally the non distortion property of ℓ^1 and c_0 (c.f. [17]) we obtain the following stronger result.

Proposition 0.21. Let X be a Banach space and $(e_n)_n$ be a ξ -order spreading model of X . Let also $E = \overline{\langle (e_n)_n \rangle}$. If the space E contains an isomorphic copy of ℓ^1 (resp. c_0) then X admits isometrically ℓ^1 (resp. c_0) as a $(\xi + 1)$ -order spreading model.

The above yield the following trichotomy.

Corollary 0.22. Let X be a reflexive space. Then one of the following holds.

- (i) The space X admits isometrically ℓ^1 as a spreading model.
- (ii) The space X admits isometrically c_0 as a spreading model.
- (iii) All spreading models of X generate reflexive spaces.

Moreover, every Schauder basic spreading model of X is unconditional.

What we have presented until now are included in Chapters 1-7. The remained chapters are devoted to several examples some of which establish the hierarchy of spreading models and others illustrate the boundaries of the corresponding theory.

An overview of the examples

We begin with an easy general method yielding ξ -order spreading models. Let $(e_n)_n$ be an 1-subsymmetric and 1-unconditional sequence in a Banach space $(E, \|\cdot\|)$, $1 \leq \xi < \omega_1$ and \mathcal{F} be a regular thin family of order ξ .

We denote by $(e_s)_{s \in \mathcal{F}}$ the natural Hamel basis of $c_{00}(\mathcal{F})$. For $x \in c_{00}(\mathcal{F})$ we set

$$\|x\|_\xi = \sup \left\{ \left\| \sum_{i=1}^l x(s_i) e_{s_i} \right\| : l \in \mathbb{N}, (s_i)_{i=1}^l \in Plm_l(\mathcal{F}) \text{ and } l \leq s_1(1) \right\}$$

and let $X_\xi = \overline{(c_{00}(\mathcal{F}), \|\cdot\|_\xi)}$ be the completion of $c_{00}(\mathcal{F})$ under the above norm.

It is easy to see that $(e_n)_n$ is an \mathcal{F} -spreading model of X_ξ generated by the \mathcal{F} -sequence $(e_s)_{s \in \mathcal{F}}$. Thus setting $A = \{e_s : s \in \mathcal{F}\}$, we have that $(e_n)_n$ belongs to the set of all spreading models of A of order ξ .

Moreover, if in addition $(e_n)_n$ is not equivalent to the usual basis of c_0 then, using Theorem 0.6, one can verify that for every $\zeta < \xi$ and every regular thin family \mathcal{G} with $o(\mathcal{G}) = \zeta$, $(e_n)_n$ does not occur as a spreading model generated by a \mathcal{G} -subsequence in A . However, it is not so immediate to construct a space X admitting the sequence $(e_n)_n$ as a ξ -order spreading model such that for every $\zeta < \xi$ the whole space X does not admit a ζ -order spreading model equivalent to $(e_n)_n$. In this direction we provide the following examples.

Theorem 0.23. For every $k \in \mathbb{N}$ there exists a reflexive space \mathfrak{X}_{k+1} with an unconditional basis $(e_s)_{s \in [\mathbb{N}]^{k+1}}$. The basis $(e_s)_{s \in [\mathbb{N}]^{k+1}}$ generates ℓ^1 as a $(k+1)$ -order spreading model and is not $(k+1)$ -Cesàro summable to any x_0 in \mathfrak{X}_{k+1} . Furthermore the space \mathfrak{X}_{k+1} does not admit a k -order ℓ^1 spreading model.

In particular the space \mathfrak{X}_{k+1} shows that the non $(k+1)$ -Cesàro summability of a $[\mathbb{N}]^{k+1}$ -sequence does not yield any further information concerning ℓ^1 spreading models of lower order.

Theorem 0.24. For every countable ordinal ξ there exists a reflexive space \mathfrak{X}_ξ with an unconditional basis satisfying the following properties:

- (i) The space \mathfrak{X}_ξ admits ℓ^1 as a ξ -order spreading model.
- (ii) For every ordinal ζ such that $\zeta + 2 < \xi$, the space \mathfrak{X}_ξ does not admit ℓ^1 as a ζ -order spreading model.

In particular, if ξ is a limit countable ordinal, then the space \mathfrak{X}_ξ does not admit ℓ^1 as a ζ -order spreading model for every $\zeta < \xi$.

The aim of the next example is to separate for $k > 1$ the class of strong k -order spreading models from the k -order ones.

Theorem 0.25. For every $1 < q < \infty$ and $k > 1$, there exists a Banach space $X_{1,q}^k$ with an unconditional basis such that $X_{1,q}^k$ admits ℓ^q as a spreading model of order k and does not admit ℓ^q as a strong l -order spreading model for every $l \in \mathbb{N}$.

The next example concerns plegma block generated ℓ^1 spreading models. Among others it shows that condition \mathcal{P} appeared in Theorem 0.16 is indeed necessary.

Theorem 0.26. There exists a reflexive space X with an unconditional basis admitting ℓ^1 as ω -order spreading model and not admitting ℓ^1 as a plegma block generated spreading model of any order.

The last example shows that the hierarchy of spreading models introduced in this work does not provide a Krivine type result (c.f. [19]). More precisely we have the following

Theorem 0.27. There exists a reflexive space X with an unconditional basis such that for every $\xi < \omega_1$ and every $(e_n)_n \in \mathcal{SM}_\xi(X)$, the space $E = \overline{\langle (e_n)_n \rangle}$ does not contain any isomorphic copy of c_0 or ℓ^p , for all $1 \leq p < \infty$.

The latter answers in the affirmative a related problem posed in [24]. Moreover it is an interesting question if the hierarchy of the spreading models of the space X is stabilized at some $\xi < \omega_1$. Furthermore by Corollary 0.22 we have that every spreading model of X generates a reflexive space not containing ℓ^p , for all $1 < p < \infty$. It is open whether all these spaces are related to reflexive spaces generated by saturation methods like Tsirelson, mixed Tsirelson and their variants.

Preliminary notation and definitions

We start with some notation related to the subsets of \mathbb{N} . As usual, we denote the set of the natural numbers by $\mathbb{N} = \{1, 2, \dots\}$. Throughout the paper we shall identify strictly increasing sequences in \mathbb{N} with their corresponding range i.e. we view every strictly increasing sequence in \mathbb{N} as a subset of \mathbb{N} and conversely every subset of \mathbb{N} as the sequence resulting from the increasing ordering of its elements. We will use capital letters as L, M, N, \dots to denote infinite subsets and lower case letters as s, t, u, \dots to denote finite subsets of \mathbb{N} .

For every infinite subset L of \mathbb{N} , $[L]^{<\infty}$ (resp. $[L]^\infty$) stands for the set of all finite (resp. infinite) subsets of L . For an $L = \{l_1 < l_2 < \dots\} \in [\mathbb{N}]^\infty$ and a positive integer $k \in \mathbb{N}$, we set $L(k) = l_k$. Similarly, for a finite subset $s = \{n_1 < \dots < n_m\}$ of \mathbb{N} and for $1 \leq k \leq m$ we set $s(k) = n_k$. Also for every nonempty $s \in [\mathbb{N}]^{<\infty}$ and $1 \leq k \leq |s|$ we set $s|k = \{s(1), \dots, s(k)\}$ and $s|0 = \emptyset$.

For an $L = \{l_1 < l_2 < \dots\} \in [\mathbb{N}]^\infty$ and a finite subset $s = \{n_1 < \dots < n_k\}$ (resp. for an infinite subset $N = \{n_1 < n_2 < \dots\}$ of \mathbb{N}), we set $L(s) = \{l_{n_1}, \dots, l_{n_k}\}$ (resp. $L(N) = \{l_{n_1}, l_{n_2}, \dots\}$).

For $s \in [\mathbb{N}]^{<\infty}$ by $|s|$ we denote the cardinality of s . For $L \in [\mathbb{N}]^\infty$ and $k \in \mathbb{N}$, we denote by $[L]^k$ the set of all $s \in [L]^{<\infty}$ with $|s| = k$. For every $s, t \in [\mathbb{N}]^{<\infty}$ we write $t < s$ if either at least one of them is the empty set, or $\max t < \min s$.

We also recall some standard notation and definitions from Banach space theory. Although the notation that we follow is the standard one, as it can be found in textbooks like [1], we present for the sake of completeness some basic concepts that are involved in this monograph.

By the term Banach space we shall always mean an infinite dimensional one. Let X be a Banach space. When we say that Z is a subspace of X we mean that Z is a closed infinite dimensional subspace of X . For a subspace Z of X , B_Z (resp. S_Z) stands for the unit ball Z , i.e. the set $\{x \in Z : \|x\| \leq 1\}$ (resp. the unit sphere of Z , i.e. the set $\{x \in Z : \|x\| = 1\}$). For a bounded linear operator $T : X \rightarrow Y$, where Y is a Banach space, we will say that T is strictly singular if there exists no subspace Z of X such that the restriction $T|_Z$ of T on Z is an isomorphic embedding.

Let $(x_n)_n$ be a sequence in X . We say that $(x_n)_n$ is bounded (resp. semi-normalized) if there exists $M > 0$ (resp. $C, c > 0$) such that $\|x_n\| \leq M$ (resp. $c \leq \|x_n\| \leq C$) for all $n \in \mathbb{N}$. We say that $(x_n)_n$ is normalized if $\|x_n\| = 1$ for all $n \in \mathbb{N}$.

A sequence $(e_n)_n$ is a Schauder basis of X if for every $x \in X$ there exists a unique sequence $(a_n)_n$ of reals such that $x = \sum_{n=1}^{\infty} a_n x_n$. As is well known the associated projections $(P_n)_n$ to $(e_n)_n$, defined by $P_n(\sum_{j=1}^{\infty} a_j e_j) = \sum_{j=1}^n a_j e_j$ for every $x = \sum_{j=1}^{\infty} a_j e_j \in X$ and $n \in \mathbb{N}$, are uniformly bounded and the quantity $\sup_n \|P_n\|$ is the basis constant of $(e_n)_n$. A sequence $(x_n)_n$ in X is called (Schauder) basic if $(x_n)_n$ forms a Schauder basis for the subspace $\overline{\langle (x_n)_n \rangle}$ of X . Two basic sequences $(x_n)_n$ and $(y_n)_n$, not necessarily belonging to the same Banach space, are called equivalent if there exist $c, C > 0$ such that for every $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$

$$c \left\| \sum_{j=1}^n a_j y_j \right\| \leq \left\| \sum_{j=1}^n a_j x_j \right\| \leq C \left\| \sum_{j=1}^n a_j y_j \right\|$$

A sequence $(x_n)_n$ in X is called C -unconditional, where $C > 0$, if for every $F \subseteq \mathbb{N}$ and every $(a_n)_n$ sequence of reals such that the series $\sum_{n=1}^{\infty} a_n x_n$ converges we have that

$$\left\| \sum_{n \in F} a_n x_n \right\| \leq C \left\| \sum_{n=1}^{\infty} a_n x_n \right\|$$

For a sequence $(x_n)_n$ in X we say that a sequence $(y_n)_n$ is a block subsequence of $(x_n)_n$ if there exist a strictly increasing sequence $(p_n)_n$ of natural numbers and a sequence $(a_n)_n$ of reals such that for every $n \in \mathbb{N}$ we have that

$$y_n = \sum_{k=p_n}^{p_{n+1}-1} a_k x_k$$

Suppose that X has a Schauder basis $(e_n)_n$. For every $x = \sum_{n=1}^{\infty} a_n e_n \in X$ we define the support of x to be the set $\text{supp}(x) = \{n \in \mathbb{N} : a_n \neq 0\}$ and we say that x is finitely supported if $\text{supp}(x)$ is finite. For $x_1, x_2 \in X$ finitely supported we write $x_1 < x_2$ if $\text{supp}(x_1) < \text{supp}(x_2)$. The biorthogonal functionals $(e_n^*)_n$ of $(e_n)_n$ are defined by $e_n^*(x) = a_n$ for every $x = \sum_{i=1}^{\infty} a_i e_i \in X$ and $n \in \mathbb{N}$. We recall that the basis $(e_n)_n$ is called shrinking if $X^* = \overline{\langle (e_n^*)_n \rangle}^{\|\cdot\|}$. The basis $(e_n)_n$ is called boundedly complete if for every sequence $(a_n)_n$ of reals such that the sequence $(\sum_{k=1}^n a_k e_k)_n$ is bounded, the series $\sum_{n=1}^{\infty} a_n e_n$ converges.

CHAPTER 1

Plegma families

In this chapter we introduce the concept of a *plegma* family of finite subsets of \mathbb{N} . Roughly speaking a plegma family is a finite sequence of finite nonempty subsets of \mathbb{N} , whose elements alternate each other in a very concrete way. This notion is the basic new ingredient for the extension of the definition of the spreading models. We also establish several Ramsey properties of plegma families that will be frequently used in the sequel. Among them there are some graph oriented concepts such as *plegma paths* and *plegma preserving* maps between thin families of finite subsets of \mathbb{N} . In the beginning of this chapter we review some classical results concerning the Ramsey theory of finite subsets of \mathbb{N} .

1. Families of finite subsets of \mathbb{N}

In this section we review some basic definitions and terminology for families of finite subsets of \mathbb{N} . This theory is traced back to the work of Nash-Williams [22] as well as to the seminal paper of P. Pudlak and V. Rodl [25] where among others, thin families of order ξ were defined for every countable ordinal ξ (see below for the relevant definitions). For more information, we refer the reader to [4] and the references therein.

1.1. Basic definitions. Recall that a family \mathcal{F} of finite subsets of \mathbb{N} is said to be *hereditary* if for every $s \in \mathcal{F}$ and $t \subseteq s$ we have that $t \in \mathcal{F}$ and *spreading* if for every $n_1 < \dots < n_k$ and $m_1 < \dots < m_k$ with $\{n_1, \dots, n_k\} \in \mathcal{F}$ and $n_1 \leq m_1, \dots, n_k \leq m_k$ we have that $\{m_1, \dots, m_k\}$ belongs to \mathcal{F} too. Also \mathcal{F} is called *compact* if the set of characteristic functions of the members of \mathcal{F} , $\{\chi_s \in \{0, 1\}^{\mathbb{N}} : s \in \mathcal{F}\}$, is a closed subspace of $\{0, 1\}^{\mathbb{N}}$. A family \mathcal{F} of finite subsets of \mathbb{N} will be called *regular* if it is compact, hereditary and spreading. For a family $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ of finite subsets of \mathbb{N} and an infinite subset $L \in [\mathbb{N}]^\omega$ we set

$$\mathcal{F} \restriction L = \{s \in \mathcal{F} : s \subseteq L\} = \mathcal{F} \cap [L]^{<\omega}$$

The *order* of a family $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ is defined as follows (see also [25]). We assign to \mathcal{F} its $(\subseteq -)$ closure

$$\widehat{\mathcal{F}} = \{t \in [\mathbb{N}]^{<\omega} : \exists s \in \mathcal{F} \text{ with } t \subseteq s\}$$

which under the initial segment ordering is a tree. If $\widehat{\mathcal{F}}$ is ill-founded (i.e. there exists an infinite sequence $(s_n)_{n \in \mathbb{N}}$ in $\widehat{\mathcal{F}}$ such that $s_n \sqsubset s_{n+1}$) then we set $o(\mathcal{F}) = \omega_1$. Otherwise, for every maximal element s of $\widehat{\mathcal{F}}$ we set $o_{\widehat{\mathcal{F}}}(s) = 0$ and recursively for every s in $\widehat{\mathcal{F}}$ we define

$$o_{\widehat{\mathcal{F}}}(s) = \sup\{o_{\widehat{\mathcal{F}}}(t) + 1 : t \in \widehat{\mathcal{F}} \text{ and } s \sqsubset t\}$$

The order of \mathcal{F} is defined to be the ordinal $o(\mathcal{F}) = o_{\widehat{\mathcal{F}}}(\emptyset)$. By convention for an empty family $\mathcal{F} = \emptyset$ we set $o(\mathcal{F}) = -1$.

For every $n \in \mathbb{N}$ we define

$$\mathcal{F}_{(n)} = \{s \in [\mathbb{N}]^{<\infty} : n < s \text{ and } \{n\} \cup s \in \mathcal{F}\}$$

where $n < s$ means that either $s = \emptyset$ or $n < \min(s)$. It follows that for every nonempty \mathcal{F}

$$(1) \quad o(\mathcal{F}) = \sup\{o(\mathcal{F}_{(n)}) + 1 : n \in \mathbb{N}\}$$

A family \mathcal{F} of finite subsets of \mathbb{N} is called *thin* if there do not exist $s \neq t$ in \mathcal{F} such that $s \sqsubseteq t$.

Definition 1.1. We will call a family \mathcal{F} of finite subsets of \mathbb{N} *regular thin* if \mathcal{F} is thin and in addition its closure $\widehat{\mathcal{F}}$ is a regular family.

Let us point out that if \mathcal{F} is a regular family, then the Cantor-Bendixson index of \mathcal{F} is equal to $o(\mathcal{F}) + 1$. Notice that a thin family \mathcal{F} is regular thin if and only if \mathcal{F} is the set of all \sqsubseteq -maximal elements of $\widehat{\mathcal{F}}$. More generally it is easy to see that for every $L \in [\mathbb{N}]^\infty$ the set of all \sqsubseteq -maximal elements of $\widehat{\mathcal{F}} \upharpoonright L$ coincides with $\mathcal{F} \upharpoonright L$. This in particular yields the following fact.

Fact 1.2. Let \mathcal{F} be a regular thin family. Then the following hold

- (i) For every $L \in [\mathbb{N}]^\infty$ we have that $\widehat{\mathcal{F} \upharpoonright L} = \widehat{\mathcal{F}} \upharpoonright L$.
- (ii) For every $n \in \mathbb{N}$, $\mathcal{F}_{(n)}$ is regular thin and $\widehat{\mathcal{F}_{(n)}} = \widehat{\mathcal{F}}_{(n)}$.
- (iii) For every $L \in [\mathbb{N}]^\infty$ we have that $o(\mathcal{F}) = o(\widehat{\mathcal{F}}) = o(\widehat{\mathcal{F}} \upharpoonright L) = o(\mathcal{F} \upharpoonright L)$.

Regular thin families as well as maps defined on them possess a central role throughout this paper. For a set X and a regular thin family \mathcal{F} , by the term \mathcal{F} -sequence in X we will understand a map $\varphi : \mathcal{F} \rightarrow X$. An \mathcal{F} -sequence in X will be usually denoted by $(x_s)_{s \in \mathcal{F}}$. More generally given $M \in [\mathbb{N}]^\infty$, a map $\varphi : \mathcal{F} \upharpoonright M \rightarrow X$ will be called an \mathcal{F} -subsequence and will be denoted by $(x_s)_{s \in \mathcal{F} \upharpoonright M}$.

1.2. Ramsey properties of families of finite subsets of \mathbb{N} . We will use the following terminology (it is also used in [13]). Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and $M \in [\mathbb{N}]^\infty$. We say that \mathcal{F} is *large* in M if for every $L \in [M]^\infty$ there exists $s \in \mathcal{F}$ such that $s \subseteq L$. We say that \mathcal{F} is *very large* in M if for every $L \in [M]^\infty$ there exists $s \in \mathcal{F}$ such that $s \sqsubseteq L$. The following is a restatement (see also [13]) of a well known theorem due to Nash-Williams [22] and F. Galvin and K. Prikry [9].

Theorem 1.3. Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and $M \in [\mathbb{N}]^\infty$. If \mathcal{F} is large in M then there exists $L \in [M]^\infty$ such that \mathcal{F} is very large in L .

Remark 1.4. If \mathcal{F} is regular thin then it is easy to see that \mathcal{F} is large in \mathbb{N} . Hence by Theorem 1.3 for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that $\mathcal{F} \upharpoonright L$ is very large in L .

The following theorem is also due to Nash-Williams [22]. Since it plays a crucial role in the sequel, for the sake of completeness we present its proof.

Theorem 1.5. Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ be a regular thin family. Then for every finite partition $\mathcal{F} = \cup_{i=1}^k \mathcal{F}_i$, ($k \geq 2$) of \mathcal{F} and every $M \in [\mathbb{N}]^\infty$ there exist $L \in [M]^\infty$ and $1 \leq i_0 \leq k$ such that $\mathcal{F} \upharpoonright L = \mathcal{F}_{i_0} \upharpoonright L$.

PROOF. It suffices to show the result only for $k = 2$ since the general case follows easily by induction. So let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ and $M \in [\mathbb{N}]^\infty$. Then either there is $L \in [M]^\infty$ such that $\mathcal{F}_1 \upharpoonright L = \emptyset$ or \mathcal{F}_1 is large in M . In the first case it is clear

that $\mathcal{F} \upharpoonright L = \mathcal{F}_2 \upharpoonright L$. In the second case by Theorem 1.3 there is $L \in [M]^\infty$ such that \mathcal{F}_1 is very large in L . We claim that $\mathcal{F} \upharpoonright L = \mathcal{F}_1 \upharpoonright L$. Indeed, let $s \in \mathcal{F} \upharpoonright L$. We choose $N \in [L]^\infty$ such that $s \sqsubseteq N$ and let $t \sqsubseteq N$ such that $t \in \mathcal{F}_1$. Then s, t are \sqsubseteq -comparable members of \mathcal{F} and since \mathcal{F} is thin they must be equal. Therefore $s = t \in \mathcal{F}_1$ and $\mathcal{F} \upharpoonright L = \mathcal{F}_1 \upharpoonright L$. \square

For two families \mathcal{F}, \mathcal{G} of finite subsets of \mathbb{N} , we write $\mathcal{F} \sqsubseteq \mathcal{G}$ (resp. $\mathcal{F} \sqsubset \mathcal{G}$) if every element in \mathcal{F} has an extension (resp. proper extension) in \mathcal{G} and every element in \mathcal{G} has an initial (resp. proper initial) segment in \mathcal{F} . The following proposition is a consequence of a more general result from [10].

Proposition 1.6. Let $\mathcal{F}, \mathcal{G} \subseteq [\mathbb{N}]^{<\infty}$ be regular thin families with $o(\mathcal{F}) < o(\mathcal{G})$. Then for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that $\mathcal{F} \upharpoonright L \sqsubset \mathcal{G} \upharpoonright L$.

PROOF. By the assertion (ii) of the above fact we have that there exists $L_1 \in [M]^\infty$ such that both \mathcal{F}, \mathcal{G} are very large in L_1 . So for every $L \in [L_1]^\infty$ and every $t \in \mathcal{G} \upharpoonright L$ there exists $s \in \mathcal{F} \upharpoonright L$ such that s, t are comparable and conversely. Let \mathcal{G}_1 be the set of all elements of \mathcal{G} which have a proper initial segment in \mathcal{F} and $\mathcal{G}_2 = \mathcal{G} \setminus \mathcal{G}_1$. By Theorem 1.5 there exist $i_0 \in \{1, 2\}$ and $L \in [L_1]^\infty$ such that $\mathcal{G} \upharpoonright L \subseteq \mathcal{G}_{i_0}$. It suffices to show that $i_0 = 1$. Indeed, if $i_0 = 2$ then for every $t \in \mathcal{G} \upharpoonright L$ there is $s \in \mathcal{F}$ such that $t \sqsubseteq s$. This in conjunction with assertion (ii) of the above fact yields that $o(\mathcal{G}) = o(\mathcal{G} \upharpoonright L) \leq o(\mathcal{F})$ which is a contradiction. \square

2. The notion of the plegma families

In this section we introduce the concept of the *plegma* families of nonempty finite subsets of \mathbb{N} . This concept possesses a fundamental role throughout the rest of the paper leading, among others, to the extension of the notion of spreading model.

Definition 1.7. Let $l \in \mathbb{N}$ and s_1, \dots, s_l be nonempty finite subsets of \mathbb{N} . The l -tuple $(s_j)_{j=1}^l$ will be called *plegma* if the following are satisfied.

- (i) For every $i, j \in \{1, \dots, l\}$ and $k \in \mathbb{N}$ with $i < j$ and $k \leq \min(|s_i|, |s_j|)$, we have that $s_i(k) < s_j(k)$.
- (ii) For every $i, j \in \{1, \dots, l\}$ and $k \in \mathbb{N}$ such that $k \leq \min(|s_i|, |s_j| - 1)$, we have that $s_i(k) < s_j(k + 1)$.

For instance a pair $(\{n_1\}, \{n_2\})$ of singletons is plegma iff $n_1 < n_2$. Also a pair of doubletons $(\{n_1, m_1\}, \{n_2, m_2\})$ is plegma iff $n_1 < n_2 < m_1 < m_2$. Moreover it is easy to see that $(s_j)_{j=1}^l$ is plegma iff for every $1 \leq k \leq l$ and $1 \leq i_1 < i_2 < \dots < i_k \leq l$ the k -tuple $(s_{i_j})_{j=1}^k$ is plegma. This in particular implies that for every $1 \leq i_1 < i_2 \leq l$ we have that $s_{i_1} \cap s_{i_2} = \emptyset$.

Notation 1.8. Let \mathcal{F} be a family of finite subsets of \mathbb{N} and $l \in \mathbb{N}$. We set

$$Plm_l(\mathcal{F}) = \{(s_j)_{j=1}^l : s_1, \dots, s_l \in \mathcal{F} \text{ and } (s_j)_{j=1}^l \text{ plegma}\}$$

Let also $Plm(\mathcal{F}) = \bigcup_{l=1}^\infty Plm_l(\mathcal{F})$.

Lemma 1.9. Let \mathcal{F} be a thin family of finite subsets of \mathbb{N} . Let $(s_j)_{j=1}^l, (t_j)_{j=1}^l$ in $Plm(\mathcal{F})$ with $|s_1| \leq \dots \leq |s_l|$, $|t_1| \leq \dots \leq |t_l|$ and $\bigcup_{j=1}^l s_j \sqsubseteq \bigcup_{j=1}^l t_j$. Then $(s_j)_{j=1}^l = (t_j)_{j=1}^l$ and hence $\bigcup_{j=1}^l s_j = \bigcup_{j=1}^l t_j$.

PROOF. Suppose that for some $1 \leq m \leq l$ we have that $(s_i)_{i < m} = (t_i)_{i < m}$. We will show that $s_m = t_m$. Let $F = \cup_{j=m}^l s_j$ and $G = \cup_{j=m}^l t_j$. Since $\cup_{j=1}^l s_j \sqsubseteq \cup_{j=1}^l t_j$, we get that $F \sqsubseteq G$. Moreover since $|s_m| \leq \dots \leq |s_l|$ and $|t_m| \leq \dots \leq |t_l|$, we easily conclude that $s_m(j) = F((j-1)(l-m+1)+1)$, for all $1 \leq j \leq |s_m|$ and similarly $t_m(j) = G((j-1)(l-m+1)+1)$, for all $1 \leq j \leq |t_m|$. Hence, as $F \sqsubseteq G$, we get that for all $1 \leq j \leq \min\{|t_m|, |s_m|\}$, $s_m(j) = t_m(j)$. Therefore s_m and t_m are \sqsubseteq -comparable which by the thinness of \mathcal{F} implies that $s_m = t_m$. By induction we have that $s_j = t_j$ for all $1 \leq j \leq l$. \square

Lemma 1.10. Let \mathcal{F} be a regular thin family. Then for every $(s_j)_{j=1}^l \in Plm(\mathcal{F})$ we have that $|s_1| \leq \dots \leq |s_l|$.

PROOF. It suffices to prove it for $l = 2$. Assume on the contrary that there exists a plegma pair (s_1, s_2) in \mathcal{F} with $|s_1| > |s_2|$. We pick $s \in [\mathbb{N}]^{<\infty}$ such that $|s| = |s_1|$, $s_2 \sqsubset s$ and $s(|s_2|+1) > \max s_1$. By the definition of plegma, we have that for every $1 \leq k \leq |s_2|$, $s_1(k) < s_2(k) = s(k)$. Hence, for every $1 \leq k \leq |s_1|$, we have that $s_1(k) \leq s(k)$. By the spreading property of $\widehat{\mathcal{F}}$ we get that $s \in \widehat{\mathcal{F}}$. Since s_2 is a proper initial segment of s we get that $s_2 \notin \mathcal{F}$, which is a contradiction. \square

Lemma 1.11. Let \mathcal{F} be a regular thin family. For every $l \in \mathbb{N}$ let

$$\mathcal{U}_l(\mathcal{F}) = \{\cup_{j=1}^l s_j : (s_j)_{j=1}^l \in Plm_l(\mathcal{F})\}$$

Then for every $l \in \mathbb{N}$ and $M \in [\mathbb{N}]^\infty$ the family $\mathcal{U}_l(\mathcal{F} \upharpoonright M)$ is thin and the map $Plm_l(\mathcal{F} \upharpoonright M) \rightarrow \mathcal{U}_l(\mathcal{F} \upharpoonright M)$ is 1-1 and onto.

PROOF. Let $(s_j)_{j=1}^l \in Plm_l(\mathcal{F} \upharpoonright M) \subseteq Plm_l(\mathcal{F})$. By Lemma 1.10 we have that $|s_1| \leq \dots \leq |s_l|$. Hence the result follows readily by Lemma 1.9. \square

Proposition 1.12. Let M be an infinite subset of \mathbb{N} , $l \in \mathbb{N}$ and \mathcal{F} be a regular thin family. Then for every finite partition $Plm_l(\mathcal{F} \upharpoonright M) = \cup_{j=1}^p A_j$, there exist $L \in [M]^\infty$ and $1 \leq j_0 \leq p$ such that $Plm_l(\mathcal{F} \upharpoonright L) \subseteq A_{j_0}$.

PROOF. Let $\mathcal{U} = \mathcal{U}_l(\mathcal{F} \upharpoonright M)$ and for $1 \leq j \leq p$, let $\mathcal{U}^{(j)} = \{\cup_{i=1}^l s_i : (s_i)_{i=1}^l \in A_j\}$. Then $\mathcal{U} = \cup_{j=1}^p \mathcal{U}^{(j)}$ and by Lemma 1.11 we have that \mathcal{U} is a thin family and $\{\mathcal{U}^{(j)}\}_{j=1}^p$ is a partition of \mathcal{U} . Hence by Theorem 1.5 there exist j_0 and $L \in [M]^\infty$ such that $\mathcal{U} \upharpoonright L \subseteq \mathcal{U}^{(j_0)}$. Since $L \in [M]^\infty$ we have that $\mathcal{U} \upharpoonright L = \mathcal{U}_l(\mathcal{F} \upharpoonright L)$. Hence $Plm_l(\mathcal{F} \upharpoonright L) \subseteq A_{j_0}$. \square

Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$, $l \in \mathbb{N}$, $M \in [\mathbb{N}]^\infty$ and $A \subseteq Plm_l(\mathcal{F})$. We will say that A is *large* in M if for every $L \in [M]^\infty$ we have that $A \cap Plm_l(\mathcal{F} \upharpoonright L) \neq \emptyset$. Under this terminology the following corollary is an immediate consequence of Proposition 1.12.

Corollary 1.13. Let \mathcal{F} be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $l \in \mathbb{N}$. Let $A \subseteq Plm_l(\mathcal{F})$ be large in M . Then for every $M' \in [M]^\infty$, there exists $L \in [M']^\infty$ such that $Plm_l(\mathcal{F} \upharpoonright L) \subseteq A$.

3. Plegma preserving maps

In this section we introduce the definition of *plegma paths* in finite subsets of \mathbb{N} . Using such paths we study maps from a regular thin family into the finite subsets of \mathbb{N} which preserve plegma pairs. The main result is Theorem 1.23 where it is proved that if \mathcal{F}, \mathcal{G} are regular thin families with $o(\mathcal{F}) < o(\mathcal{G})$ then there is no plegma preserving map from \mathcal{F} to \mathcal{G} .

3.1. Plegma paths of finite subsets of \mathbb{N} .

Definition 1.14. Let $k \in \mathbb{N}$ and s_0, \dots, s_k be nonempty finite subsets of \mathbb{N} . We will say that $(s_j)_{j=0}^k$ is a *plegma path of length k from s_0 to s_k* , if for every $0 \leq j \leq k-1$, the pair (s_j, s_{j+1}) is plegma. Similarly a sequence $(s_j)_{j \in \mathbb{N}}$ of nonempty finite subsets of \mathbb{N} will be called an infinite plegma path if for every $j \in \mathbb{N}$ the pair (s_j, s_{j+1}) is plegma.

Lemma 1.15. Let s_0, s be two nonempty finite subsets of \mathbb{N} such that $s_0 < s$. Let (s_0, \dots, s_{k-1}, s) be a plegma path of length k from s_0 to s . Then

$$k \geq \min\{|s_i| : 0 \leq i \leq k-1\}$$

PROOF. Suppose that $k < \min\{|s_i| : 0 \leq i \leq k-1\}$. Then $s(1) < s_{k-1}(2) < s_{k-2}(3) < \dots < s_1(k) < s_0(k+1)$, which contradicts that $s_0 < s$. \square

For a family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ a *plegma path in \mathcal{F}* is a (finite or infinite) plegma path consisted of elements of \mathcal{F} . It is easy to verify the existence of infinite plegma paths in \mathcal{F} whenever \mathcal{F} is very large in an infinite subset L of \mathbb{N} . In particular let $s \in \mathcal{F} \restriction L$ satisfying the next property: for every $j = 1, \dots, |s| - 1$ there exists $l \in L$ such that $s(j) < l < s(j+1)$. Then it is straightforward that there exists $s' \in \mathcal{F} \restriction L$ such that the pair (s, s') is plegma and moreover s' shares the same property with s . Based on this one can built an infinite plegma path in \mathcal{F} consisted of elements having the above property. These remarks motivate the following definition.

Definition 1.16. Let \mathcal{F} be a family of finite subsets of \mathbb{N} and $L \in [\mathbb{N}]^\infty$. We define the *skipped restriction of \mathcal{F} in L* to be the family

$$\mathcal{F} \restriction L = \left\{ s \in \mathcal{F} \restriction L : \forall j = 1, \dots, |s| - 1, \exists l \in L \text{ such that } s(j) < l < s(j+1) \right\}$$

Notice that if \mathcal{F} is a regular thin family and $L \in [\mathbb{N}]^\infty$ such that $\mathcal{F} \restriction L$ is very large in L , then

$$\mathcal{F} \restriction L = \left\{ s \in \mathcal{F} \restriction L : \exists s' \in \mathcal{F} \restriction L \text{ such that } (s, s') \text{ is plegma} \right\}$$

Also, by Lemma 1.10, we have that the lengths of the elements of any plegma path in \mathcal{F} forms an non decreasing sequence. This property is a key ingredient of the proof of the next proposition.

Proposition 1.17. Let \mathcal{F} be a regular thin family and $L \in [\mathbb{N}]^\infty$ such that \mathcal{F} is very large in L . Then for every $s_0, s \in \mathcal{F} \restriction L$ with $s_0 < s$ there exists a plegma path (s_0, \dots, s_{k-1}, s) in $\mathcal{F} \restriction L$ of length $k = |s_0|$ from s_0 to s . Moreover $k = |s_0|$ is the minimal length of a plegma path in $\mathcal{F} \restriction L$ from s_0 to s .

PROOF. We will actually prove the following stronger result. For every t in the closure $\widehat{\mathcal{F} \restriction L}$ of $\mathcal{F} \restriction L$ and $s \in \mathcal{F} \restriction L$ with $t < s$ there exists an plegma path of length $|t|$ from t to s such that all its elements except t belong to $\mathcal{F} \restriction L$. The proof will be done by induction on the length of t . The case $|t| = 1$ is trivial, since (t, s) is already a plegma path of length 1 from t to s . Suppose that for some $k \in \mathbb{N}$ the above holds for all t in $\widehat{\mathcal{F} \restriction L}$ with $|t| = k$. Let $t \in \widehat{\mathcal{F} \restriction L}$ with $|t| = k+1$ and $s \in \mathcal{F} \restriction L$ with $t < s$. Then there exist $n_1 < n_2 < \dots < n_{k+1}$ in \mathbb{N} such that $t = \{L(n_j) : 1 \leq j \leq k+1\}$. We set $t_0 = \{L(n_j - 1) : 2 \leq j \leq k+1\}$. Notice that by the spreading property of $\widehat{\mathcal{F}}$ the element t_0 belongs to $\widehat{\mathcal{F}}$. Since \mathcal{F} is very large in L , we actually have that $t_0 \in \widehat{\mathcal{F} \restriction L}$. Thus by the inductive hypothesis there exists a plegma path $(t_0, s_1, \dots, s_{k-1}, s)$ of length k from t_0 to s

s with $s_1, \dots, s_{k-1}, s \in \mathcal{F} \restriction L$. Let $l = |s_1|$ and $m_1 < \dots < m_l$ such that $s_1 = \{L(m_j) : 1 \leq j \leq l\}$. Again by the spreading property of \mathcal{F} it is easy to see the following

- (i) $|s_1| \geq |t|$, that is $l \geq k + 1$.
- (ii) $t_0 \in \overline{\mathcal{F}} \restriction \overline{L} \setminus \mathcal{F} \restriction L$.
- (iii) There exists a (unique) proper extension s_0 of t_0 such that $s_0 \in \mathcal{F} \restriction L$ and $s_0 \subseteq t_0 \cup \{L(m_j - 1) : k + 1 \leq j \leq l\}$.

It is easy to check that (t, s_0) and (s_0, s_1) are plegma pairs. Hence the sequence $(t, s_0, \dots, s_{k-1}, s)$ is a plegma path of length $k + 1$ from t to s with $s_0, \dots, s_{k-1}, s \in \mathcal{F} \restriction L$ and the proof is complete.

Finally by Lemmas 1.15 and 1.10, every plegma path in \mathcal{F} from s_0 to s is of length at least $|s_0|$. Therefore the path (s_0, \dots, s_{k-1}, s) is of minimal length. \square

Remark 1.18. In terms of graph theory the above proposition states that in the directed graph with vertices the elements of $\mathcal{F} \restriction L$ and edges the plegma pairs (s, t) in $\mathcal{F} \restriction L$, the distance between two vertices s_0 and s with $s_0 < s$ is equal to the cardinality of s_0 .

3.2. Plegma preserving maps between thin families.

Definition 1.19. Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and $\varphi : \mathcal{F} \rightarrow [\mathbb{N}]^{<\infty}$. We will say that the map φ is *plegma preserving* if for every plegma pair (s_1, s_2) in \mathcal{F} , either $(\varphi(s_1), \varphi(s_2))$ or $(\varphi(s_2), \varphi(s_1))$ is plegma.

A map $\varphi : \mathcal{F} \rightarrow [\mathbb{N}]^{<\infty}$ will be called *plegma monotone* if for every plegma pair (s_1, s_2) in \mathcal{F} , the pair $(\varphi(s_1), \varphi(s_2))$ is also plegma.

Finally a plegma monotone map $\varphi : \mathcal{F} \rightarrow [\mathbb{N}]^{<\infty}$ will be called *plegma canonical* if in addition it satisfies the following:

- (i) For every plegma pair (s_1, s_2) in \mathcal{F} we have that $|\varphi(s_1)| \leq |\varphi(s_2)|$.
- (ii) For every $s \in \mathcal{F}$ we have that $|\varphi(s)| \leq |s|$.

Proposition 1.20. Let \mathcal{F} be a regular thin family and $\varphi : \mathcal{F} \rightarrow [\mathbb{N}]^{<\infty}$ a plegma preserving map. Then for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that the restriction $\varphi|_{\mathcal{F} \restriction L}$ of φ on $\mathcal{F} \restriction L$ is plegma canonical.

PROOF. By Proposition 1.12 there exists $L_0 \in [M]^\infty$ such that either $\varphi|_{\mathcal{F} \restriction L_0}$ is plegma monotone, or for every $(s_1, s_2) \in Plm_2(\mathcal{F} \restriction L_0)$ the pair $(\varphi(s_2), \varphi(s_1))$ is plegma. If the second alternative holds, then for an infinite plegma path $(s_k)_{k \in \mathbb{N}}$ in $\mathcal{F} \restriction L_0$, the sequence $(\min(s_k))_{k \in \mathbb{N}}$ would be a strictly decreasing sequence of natural numbers, which is impossible. Hence the restriction $\varphi|_{\mathcal{F} \restriction L_0}$ is plegma monotone.

Using Proposition 1.12 once more we get $L_1 \in [L_0]^\infty$ such that either for every plegma pair (s_1, s_2) in $\mathcal{F} \restriction L_1$, $|\varphi(s_1)| \leq |\varphi(s_2)|$, or for every plegma pair (s_1, s_2) in $\mathcal{F} \restriction L_1$, $|\varphi(s_1)| > |\varphi(s_2)|$. We will show that the second alternative leads to a contradiction. Indeed, suppose on the contrary for every plegma pair (s_1, s_2) in $\mathcal{F} \restriction L_1$, $|\varphi(s_1)| > |\varphi(s_2)|$. Let $(s_n)_{n \in \mathbb{N}}$ be an infinite plegma path in $\mathcal{F} \restriction L_1$. Then the sequence $(|s_n|)_{n \in \mathbb{N}}$ forms a strictly decreasing sequence of natural numbers, which is impossible.

Reapplying Proposition 1.12 there exists $L_2 \in [L_1]^\infty$ such that either $|\varphi(s)| \leq |s|$, for all $s \in \mathcal{F} \restriction L_2$, or $|\varphi(s)| > |s|$, for all $s \in \mathcal{F}$. The second alternative is excluded as follows. Firstly, we may assume that \mathcal{F} is very large in L_2 . Let $(s_j)_{j \geq 0}$ be an infinite plegma path in $\mathcal{F} \restriction L_2$. As we have already seen, for every $j \geq 0$ we

have that $s_j \in \mathcal{F} \upharpoonright L_2$. Since the sequences $(\min s_j)_{j \geq 0}$ and $(\min \varphi(s_j))_{j \geq 0}$ are strictly increasing, there exists $j_0 \in \mathbb{N}$ such that $s_0 < s_{j_0}$ and $\varphi(s_0) < \varphi(s_{j_0})$. Let $k_0 = |s_0|$. Then by Proposition 1.17 there exists a plegma path $(t_i)_{i=0}^{k_0}$ in $\mathcal{F} \upharpoonright L_2$ with $t_0 = s_0$ and $t_{k_0} = s_{j_0}$. Since φ restricted to $\mathcal{F} \upharpoonright L_1$ is plegma monotone, we have that $(\varphi(t_i))_{i=0}^{k_0}$ is also a plegma path of length k_0 from $\varphi(t_0)$ to $\varphi(t_{k_0})$. Hence by Lemma 1.15 we should have $\min\{|\varphi(t_i)| : 0 \leq i \leq k_0 - 1\} \leq k_0$, which is impossible since $|\varphi(t_i)| > |t_i| \geq k_0$. \square

Definition 1.21. Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and $L \in [\mathbb{N}]^\infty$. We define

$$\mathcal{F}(L^{-1}) = \left\{ t \in [\mathbb{N}]^{<\infty} : L(t) \in \mathcal{F} \right\}$$

It is easy to see that for every family \mathcal{F} of finite subsets of \mathbb{N} and $L \in [\mathbb{N}]^\infty$ the following hold

- (a) If \mathcal{F} is very large in L , then the family $\mathcal{F}(L^{-1})$ is very large in \mathbb{N} .
- (b) If \mathcal{F} is regular thin then so does the family $\mathcal{F}(L^{-1})$.
- (c) $o(\mathcal{F}(L^{-1})) = o(\mathcal{F} \upharpoonright L)$. In particular if \mathcal{F} is regular thin then $o(\mathcal{F}(L^{-1})) = o(\mathcal{F})$.

Lemma 1.22. Let \mathcal{F} be a regular thin family and $\varphi : \mathcal{F} \rightarrow [\mathbb{N}]^{<\infty}$ a plegma preserving map. Then for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that if $\psi : \mathcal{F}(L^{-1}) \rightarrow [\mathbb{N}]^{<\infty}$ is defined by $\psi(t) = \varphi(L(t))$ for every $t \in \mathcal{F}(L^{-1})$, then the following are satisfied:

- (a) The map ψ is plegma canonical.
- (b) For every $t \in \mathcal{F}(L^{-1})$ and every $i \leq |\psi(t)|$, we have that $\psi(t)(i) > t(i)$.

PROOF. By Proposition 1.20 there exists $L_1 \in [M]^\infty$ such that the restriction $\varphi|_{\mathcal{F} \upharpoonright L_1}$ of φ on $\mathcal{F} \upharpoonright L_1$ is plegma canonical. We may assume that \mathcal{F} is very large in L_1 . Let $\psi_1 = \varphi \circ L_1 : \mathcal{F}(L_1^{-1}) \rightarrow [\mathbb{N}]^{<\infty}$. It is easy to check that ψ_1 is plegma canonical.

Claim: For every $u \in \mathcal{F}(L_1^{-1}) \upharpoonright \mathbb{N}$ and $1 \leq i \leq |\psi_1(u)|$, we have $u(i) \leq \psi_1(u)(i)$.

PROOF OF CLAIM. We will show that for every $i \in \mathbb{N}$ the following holds: for every $u \in \mathcal{F}(L_1^{-1}) \upharpoonright \mathbb{N}$ with $i \leq |\psi_1(u)|$, $u(i) \leq \psi_1(u)(i)$. Indeed, let $i = 1$ and let $u \in \mathcal{F}(L_1^{-1}) \upharpoonright \mathbb{N}$. If $u(1) = 1$ then obviously $\psi_1(u)(1) \geq 1 = u(1)$. Suppose that for some $k \in \mathbb{N}$ and every $u' \in \mathcal{F}(L_1^{-1}) \upharpoonright \mathbb{N}$ with $u'(1) = k$ we have that $\psi_1(u')(1) \geq k$.

Let $u \in \mathcal{F}(L_1^{-1}) \upharpoonright \mathbb{N}$ with $u(1) = k + 1$. Since $\mathcal{F}(L_1^{-1})$ is regular and very large in \mathbb{N} there exists a unique $u' \in \mathcal{F}(L_1^{-1})$ with $u' \sqsubseteq u - 1 = \{u(i) - 1 : 1 \leq i \leq |u|\}$. Then (u', u) is an plegma pair and $u'(1) = k$. Since ψ_1 is plegma canonical we have that $(\psi_1(u'), \psi_1(u))$ is plegma. Hence $\psi_1(u)(1) > \psi_1(u')(1) \geq u'(1) = k$, that is $\psi_1(u)(1) \geq k + 1 = u(1)$. By induction on $k = u(1)$, we get that for all $u \in \mathcal{F}(L_1^{-1}) \upharpoonright \mathbb{N}$, $u(1) \leq \psi_1(u)(1)$.

Suppose now that for some $i \in \mathbb{N}$, it holds that for every $u' \in \mathcal{F}(L_1^{-1}) \upharpoonright \mathbb{N}$ with $i \leq |\psi_1(u')|$, $u'(i) \leq \psi_1(u')(i)$. Let $u \in \mathcal{F}(L_1^{-1}) \upharpoonright \mathbb{N}$ with $i + 1 \leq |\psi_1(u)|$. Clearly there exists $u' \in \mathcal{F}(L_1^{-1}) \upharpoonright \mathbb{N}$ such that $\{u(\rho) - 1 : 2 \leq \rho \leq |u|\} \sqsubseteq u'$. Then (u, u') is plegma, $|u| \leq |u'|$ and $u(i + 1) = u'(i) + 1$. Since ψ_1 is plegma canonical, $(\psi_1(u), \psi_1(u'))$ is plegma and $i + 1 \leq |\psi_1(u)| \leq |\psi_1(u')|$. Hence, $\psi_1(u)(i + 1) > \psi_1(u')(i) \geq u'(i) = u(i + 1) - 1$, that is $\psi_1(u)(i + 1) \geq u(i + 1)$. By induction on $i \in \mathbb{N}$ the proof of the claim is complete. \square

Let $N_0 = \{2\rho : \rho \in \mathbb{N}\}$ and $L = L_1(N_0)$. It is easy to check that for every $t \in \mathcal{F}(L^{-1})$, $N_0(t) = 2t = \{2t(i) : 1 \leq i \leq |t|\}$ and $N_0(t) \in \mathcal{F}(L_1^{-1}) \upharpoonright \mathbb{N}$. Let $\psi = \varphi \circ L$. Then $\psi = \psi_1 \circ N_0$. Indeed for every $t \in \mathcal{F}(L^{-1})$

$$\psi(t) = \varphi(L(t)) = \varphi(L_1(N_0(t))) = \psi_1(N_0(t))$$

Then for every $t \in \mathcal{F}(L^{-1})$ and $1 \leq i \leq |\psi(t)|$ we have that $\psi(t)(i) = \psi_1(N_0(t))(i) \geq N_0(t)(i) = 2t(i) > t(i)$. \square

Theorem 1.23. Let \mathcal{F}, \mathcal{G} be regular thin families. If $o(\mathcal{F}) < o(\mathcal{G})$ then there is no plegma preserving map from \mathcal{F} to \mathcal{G} . Precisely for every $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ and $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that for every (s_1, s_2) plegma pair in $\mathcal{F} \upharpoonright L$ neither $(\phi(s_1), \phi(s_2))$ nor $(\phi(s_2), \phi(s_1))$ is plegma.

PROOF. Assume on the contrary that there exists $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ plegma preserving map. By Lemma 1.22 there exists $L \in [\mathbb{N}]^\infty$ such that if $\psi : \mathcal{F}(L^{-1}) \rightarrow \mathcal{G}$ is defined by $\psi(t) = \varphi(L(t))$ for every $t \in \mathcal{F}(L^{-1})$, then the following are satisfied:

- (a) The map ψ is plegma canonical.
- (b) For every $t \in \mathcal{F}(L^{-1})$ and every $i \leq |\psi(t)|$, we have that $\psi(t)(i) > t(i)$.

Since $o(\mathcal{F}(L^{-1})) = o(\mathcal{F}) < o(\mathcal{G})$, by Proposition 1.6 we have that there exists $N \in [\mathbb{N}]^\infty$ such that

$$(2) \quad \mathcal{F}(L^{-1}) \upharpoonright N \sqsubset \mathcal{G} \upharpoonright N$$

We are now ready to derive a contradiction. Indeed, pick $t_0 \in \mathcal{F}(L^{-1}) \upharpoonright N$. Then $t_0 \in \widehat{\mathcal{G}} \setminus \mathcal{G}$. But then by (2) and the spreading property of $\widehat{\mathcal{G}}$ we should have $\psi(t_0) \in \widehat{\mathcal{G}} \setminus \mathcal{G}$ which is impossible. \square

Remark 1.24. As we will show in Corollary 2.12, if $o(\mathcal{F}) \leq o(\mathcal{G})$ then for every $M \in [\mathbb{N}]^\infty$ there exist $N \in [\mathbb{N}]^\infty$ and a plegma canonical map $\varphi : \mathcal{G} \upharpoonright N \rightarrow \mathcal{F} \upharpoonright M$.

CHAPTER 2

The hierarchy of spreading models

In this chapter we will introduce the notion of higher order spreading model, which is a generalization of the classical one defined by A. Brunel and L. Sucheston in [5]. The main new ingredient here is the concept of an \mathcal{F} -sequence in a Banach space, where \mathcal{F} is a regular thin family. The \mathcal{F} -sequences generate the \mathcal{F} -spreading models in a similar way as the usual sequences generate the classical spreading models in a Banach space X . The \mathcal{F} -sequence and the corresponding spreading model are connected through the concept of plegma. The proof of the existence of \mathcal{F} -spreading models follows similar steps as the standard one of the classical case and heavily relies on the Ramsey properties of plegma families stated in Section 2. In the second subsection we provide a classification of higher order spreading models. More precisely we show that if a spreading model is generated by an \mathcal{F} -sequence in a Banach space X , then it is also generated by an \mathcal{F}' -sequence in X , where \mathcal{F}' is any other regular thin family with the same order as \mathcal{F} . This explains that a spreading model generated by an \mathcal{F} -sequence is completely determined by the order of the regular thin family \mathcal{F} . As consequence we provide a classification of all spreading models.

1. Definition and existence of \mathcal{F} -spreading models

Let X be a Banach space and let \mathcal{F} be a regular thin family. An \mathcal{F} -sequence $(x_s)_{s \in \mathcal{F}}$ in X will be called *bounded* (resp. *seminormalized*) if there exists $C > 0$ (resp. $0 < c < C$) such that $\|x_s\| \leq C$ (resp. $c \leq \|x_s\| \leq C$) for every $s \in \mathcal{F}$.

We fix for the sequel a regular thin family \mathcal{F} and a bounded \mathcal{F} -sequence $(x_s)_{s \in \mathcal{F}}$ in a space X .

Lemma 2.1. Let $l \in \mathbb{N}$, $N \in [\mathbb{N}]^\infty$ and $\delta > 0$. Then there exists $L \in [N]^\infty$ such that

$$\left| \left\| \sum_{j=1}^l a_j x_{t_j} \right\| - \left\| \sum_{j=1}^l a_j x_{s_j} \right\| \right| \leq \delta$$

for every $(t_j)_{j=1}^l, (s_j)_{j=1}^l \in Plm_l(\mathcal{F} \upharpoonright L)$ and $a_1, \dots, a_l \in [-1, 1]$.

PROOF. Let $(\mathbf{a}_j)_{j=1}^{n_0}$ be a $\frac{\delta}{3l}$ -net of the unit ball of \mathbb{R}^l with $\|\cdot\|_\infty$. We set $N_0 = N$. By a finite induction on $1 \leq k \leq n_0$, we construct a decreasing sequence $N_0 \supseteq N_1 \supseteq \dots \supseteq N_{n_0}$ as follows. Suppose that N_0, \dots, N_{k-1} have been constructed. Define $g_k : Plm_l(\mathcal{F} \upharpoonright N_{k-1}) \rightarrow [0, lC]$ such that $g_k((s_j)_{j=1}^l) = \left\| \sum_{j=1}^l a_j^k x_{s_j} \right\|$, where $\mathbf{a}_k = (a_j^k)_{j=1}^l$. By diving the interval $[0, lC]$ into disjoint intervals of length $\frac{\delta}{3}$, Proposition 1.12 yields a $N_k \in [N_{k-1}]^\infty$ such that for every $(t_j)_{j=1}^l, (s_j)_{j=1}^l \in Plm_l(\mathcal{F} \upharpoonright N_k)$, we have $|g_k((t_j)_{j=1}^l) - g_k((s_j)_{j=1}^l)| < \frac{\delta}{3}$. Proceeding in this way we conclude that for every $(s_j)_{j=1}^l, (t_j)_{j=1}^l \in Plm_l(\mathcal{F} \upharpoonright N_{n_0})$ and $1 \leq k \leq n_0$ we have

that

$$\left| \left\| \sum_{j=1}^l a_j^k x_{t_j} \right\| - \left\| \sum_{j=1}^l a_j^k x_{s_j} \right\| \right| \leq \frac{\delta}{3}$$

Taking into account that $(\mathbf{a}_j)_{j=1}^{n_0}$ is a $\frac{\delta}{3}$ -net of the unit ball of $(\mathbb{R}^l, \|\cdot\|_\infty)$ it is easy to see that $L = N_{n_0}$ is as desired. \square

Using standard diagonalization arguments we get the following.

Proposition 2.2. Let \mathcal{F} be a regular thin family, $(\delta_n)_n$ be a decreasing null sequence of positive real numbers and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in a Banach space X with $\|x_s\| \leq C$, for some constant $C > 0$. Then for every $N \in [\mathbb{N}]^\infty$ there exists $M \in [\mathbb{N}]^\infty$ such that for every $1 \leq k \leq l \in \mathbb{N}$, $(t_j)_{j=1}^k, (s_j)_{j=1}^k \in Plm_k(\mathcal{F} \upharpoonright M)$ with $s_1(1), t_1(1) \geq M(l)$ and $a_1, \dots, a_k \in [-1, 1]$, we have that

$$\left| \left\| \sum_{j=1}^k a_j x_{t_j} \right\| - \left\| \sum_{j=1}^k a_j x_{s_j} \right\| \right| \leq \delta_l$$

Hence for every $l \in \mathbb{N}$ and $a_1, \dots, a_l \in \mathbb{R}$ and every sequence $((s_j^n)_{j=1}^l)_n$ of plegma l -tuples in $\mathcal{F} \upharpoonright M$ with $s_1^n(1) \rightarrow \infty$ the sequence $(\|\sum_{j=1}^l a_j x_{s_j^n}\|)_{n \in \mathbb{N}}$ is Cauchy, with the limit independent from the choice of the sequence $((s_j^n)_{j=1}^l)_{n \in \mathbb{N}}$.

In particular there exists a seminorm $\|\cdot\|_*$ on $c_{00}(\mathbb{N})$ under which the natural Hamel basis $(e_n)_n$ is a 1-subsymmetric sequence and

$$\left| \left\| \sum_{j=1}^k a_j x_{s_j} \right\| - \left\| \sum_{j=1}^k a_j e_j \right\|_* \right| \leq \delta_l$$

Let us notice that there do exist bounded \mathcal{F} -sequences in Banach spaces such that no seminorm resulting from Proposition 2.2 is a norm. Also we should point out that even if the $\|\cdot\|_*$ is a norm the sequence $(e_n)_{n \in \mathbb{N}}$ is not necessarily Schauder basic. In the sequel we shall give sufficient conditions for the seminorm to be a norm and later for the sequence $(e_n)_{n \in \mathbb{N}}$ to be a Schauder basic or even an unconditional one.

Definition 2.3. Let X be a Banach space, \mathcal{F} be a regular thin family, $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in X and $M \in [\mathbb{N}]^\infty$. Let $(E, \|\cdot\|_*)$ be an infinite dimensional seminormed linear space with Hamel basis $(e_n)_{n \in \mathbb{N}}$.

We will say that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model if the following is satisfied. There exists a null sequence $(\delta_n)_{n \in \mathbb{N}}$ of positive reals such that

$$\left| \left\| \sum_{j=1}^k a_j x_{s_j} \right\| - \left\| \sum_{j=1}^k a_j e_j \right\|_* \right| \leq \delta_l$$

for every $1 \leq k \leq l$, every plegma k -tuple $(s_j)_{j=1}^k \in Plm_k(\mathcal{F} \upharpoonright M)$ with $s_1(1) \geq M(l)$ and every choice of $a_1, \dots, a_k \in [-1, 1]$.

We will also say that $(x_s)_{s \in \mathcal{F}}$ admits $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model if there exists $M \in [\mathbb{N}]^\infty$ such that the subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model.

Finally, for a subset A of X , we will say that $(e_n)_{n \in \mathbb{N}}$ is an \mathcal{F} -spreading model of A if there exists an \mathcal{F} -sequence $(x_s)_{s \in \mathcal{F}}$ in A which admits $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model.

The next remark is straightforward by the above definition.

Remark 2.4. Let \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in a Banach space X . Let $M \in [\mathbb{N}]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model. Then the following hold.

- (i) The sequence $(e_n)_{n \in \mathbb{N}}$ is 1-subsymmetric, i.e. for every $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}$ and $k_1 < \dots < k_n$ in \mathbb{N} , $\|\sum_{j=1}^n a_j e_j\|_* = \|\sum_{j=1}^n a_j e_{k_j}\|_*$.
- (ii) For every $M' \in [M]^\infty$ we have that $(x_s)_{s \in \mathcal{F} \upharpoonright M'}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model.
- (iii) For every $(\delta_n)_{n \in \mathbb{N}}$ null sequence of positive reals there exists $M' \in [M]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright M'}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model with respect to $(\delta_n)_{n \in \mathbb{N}}$.
- (iv) If $\|\cdot\|, \|\cdot\|$ are two equivalent norms on X , then every \mathcal{F} -spreading model admitted by $(X, \|\cdot\|)$ is equivalent an \mathcal{F} -spreading model admitted by $(X, \|\cdot\|)$.

Remark 2.5. Let $(x_s)_{s \in [\mathbb{N}]^k}$ be an $[\mathbb{N}]^k$ -sequence in a Banach space X and $M \in [\mathbb{N}]$. We set $y_s = x_{M(s)}$, for all $s \in [\mathbb{N}]^k$. Then it is straightforward that the $[\mathbb{N}]^k$ -subsequence $(x_s)_{s \in [M]^k}$ generates an $[\mathbb{N}]^k$ -spreading model $(e_n)_{n \in \mathbb{N}}$ iff the $[\mathbb{N}]^k$ -sequence $(y_s)_{s \in [\mathbb{N}]^k}$ generates $(e_n)_{n \in \mathbb{N}}$ as an $[\mathbb{N}]^k$ -spreading model.

Remark 2.6. Let X be a Banach space, \mathcal{F} be a regular thin family, $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in X and $M \in [\mathbb{N}]^\infty$. Let $(E, \|\cdot\|_*)$ be an infinite dimensional seminormed linear space with Hamel basis $(e_n)_{n \in \mathbb{N}}$. Assume that there exists a null sequence $(\delta_n)_{n \in \mathbb{N}}$ of positive reals such that

$$\left| \left\| \sum_{j=1}^l a_j x_{s_j} \right\| - \left\| \sum_{j=1}^l a_j e_j \right\|_* \right| \leq \delta_l$$

for every $l \in \mathbb{N}$, every plegma l -tuple $(s_j)_{j=1}^l \in Plm_l(\mathcal{F} \upharpoonright M)$ with $s_1(1) \geq M(l)$ and every choice of $a_1, \dots, a_l \in [-1, 1]$. Then we may pass to an $L \in [M]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model. It is enough to set $L = \{M(1 + \sum_{j=1}^p j - 1) : p \in \mathbb{N}\}$. This set has the property that for every $l \in \mathbb{N}$ there exist $l-1$ elements of M between $L(l)$ and $L(l+1)$. So every plegma k -tuple after $L(l)$, with $1 \leq k \leq l$, in $\mathcal{F} \upharpoonright L$ can be extended to a plegma l -tuple after $L(l) \geq M(l)$ in $\mathcal{F} \upharpoonright M$.

Under the Definition 2.3 we get the following reformulation of Proposition 2.2.

Theorem 2.7. Let \mathcal{F} be a regular thin family and X a Banach space. Then every bounded \mathcal{F} -sequence in X admits an \mathcal{F} -spreading model.

In particular for every bounded \mathcal{F} -sequence $(x_s)_{s \in \mathcal{F}}$ in X and every $N \in [\mathbb{N}]^\infty$ there exists $M \in [N]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates \mathcal{F} -spreading model.

2. Spreading models of order ξ

We start with some combinatorial properties of regular families.

2.0.1. *Embeddings of regular families and plegma preserving maps.* Let $L \in [\mathbb{N}]^\infty$. We define

$$\mathcal{F}(L) = \{L(s) : s \in \mathcal{F}\}$$

Notice that $o(\mathcal{F}) = o(\mathcal{F}(L))$ and if \mathcal{F} is compact (resp. hereditary) then $\mathcal{F}(L)$ is also compact (resp. hereditary). We will need the following easily verified lemma concerning the structure of families of the form $\mathcal{F}(L)$ for a spreading family \mathcal{F} .

Lemma 2.8. Let \mathcal{F} be a spreading family of finite subsets of \mathbb{N} . Then

- (i) For every $L_1 \subseteq L_2$ in $[\mathbb{N}]^\infty$, we have that $\mathcal{F}(L_1) \subseteq \mathcal{F}(L_2)$.
- (ii) For every $k \in \mathbb{N}$, $L_1, L_2 \in [\mathbb{N}]^\infty$ with $\{L_1(j) : j > k\} \subseteq \{L_2(j) : j > k\}$, we have that $\mathcal{F}_{(k)}(L_1) \subseteq \mathcal{F}_{(k)}(L_2)$ (where $\mathcal{F}_{(k)}(L) = \{L(s) : s \in \mathcal{F}_{(k)}\}$).

Theorem 2.9. Let \mathcal{F}, \mathcal{G} be regular families of finite subsets of \mathbb{N} with $o(\mathcal{F}) \leq o(\mathcal{G})$. Then for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that $\mathcal{F}(L) \subseteq \mathcal{G}$.

PROOF. If $o(\mathcal{F}) \leq 0$ then the conclusion trivially holds. Suppose that for some $\xi < \omega_1$ the proposition is true for every regular families $\mathcal{F}', \mathcal{G}' \subseteq [\mathbb{N}]^{<\infty}$ such that $o(\mathcal{F}') < \xi$ and $o(\mathcal{F}') \leq o(\mathcal{G}')$. Let \mathcal{F}, \mathcal{G} be regular with $o(\mathcal{F}) = \xi$ and let $M \in [\mathbb{N}]^\infty$. By (1) we have that $o(\mathcal{F}_{(1)}) < o(\mathcal{F})$. Hence $o(\mathcal{F}_{(1)}) < o(\mathcal{G})$ and so again by (1) there is some $l_1 \in \mathbb{N}$ such that $o(\mathcal{F}_{(1)}) < o(\mathcal{G}_{(l_1)})$. Since \mathcal{G} is spreading we have that $o(\mathcal{G}_{(l_1)}) \leq o(\mathcal{G}_{(n)})$ for all $n \geq l_1$ and therefore we may suppose that $l_1 \in M$. It is easy to see that $\mathcal{F}_{(1)}$ and $\mathcal{G}_{(l_1)}$ are regular families. Hence by our inductive hypothesis there is $L_1 \in [M]^\infty$ such that $\mathcal{F}_{(1)}(L_1) \subseteq \mathcal{G}_{(l_1)}$.

Proceeding in the same way we construct a strictly increasing sequence $l_1 < l_2 < \dots$ in M and a decreasing sequence $M = L_0 \supset L_1 \supset \dots$ of infinite subsets of M such that for all $j \geq 1$, the following properties are satisfied.

- (i) $l_{j+1} \in L_j$.
- (ii) $l_{j+1} > L_j(j)$.
- (iii) $\mathcal{F}_{(j)}(L_j) \subseteq \mathcal{G}_{(l_j)}$.

We set $L = \{l_j\}_{j \in \mathbb{N}}$. We claim that $\mathcal{F}(L) \subseteq \mathcal{G}$. Indeed, by the above construction we have that for every $k \in \mathbb{N}$, $\{L(j)\}_{j > k} \subseteq \{L_k(j)\}_{j > k}$. Therefore by the second part of Lemma 2.8 and the third condition of the construction we get that

$$(3) \quad \mathcal{F}_{(k)}(L) \subseteq \mathcal{F}_{(k)}(L_k) \subseteq \mathcal{G}_{(l_k)}$$

It is easy to see that $\mathcal{F}_{(k)}(L) = \mathcal{F}(L)_{(l_k)}$ and so by (3) we have that $\mathcal{F}(L)_{(l_k)} \subseteq \mathcal{G}_{(l_k)}$. Since this holds for every $k \in \mathbb{N}$, we conclude that $\mathcal{F}(L) \subseteq \mathcal{G}$. \square

Corollary 2.10. Let \mathcal{F}, \mathcal{G} be regular families of finite subsets of \mathbb{N} with $o(\mathcal{F}) = o(\mathcal{G})$. Then for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that $\mathcal{F}(L) \subseteq \mathcal{G}$ and $\mathcal{G}(L) \subseteq \mathcal{F}$.

Proposition 2.11. Let \mathcal{F}, \mathcal{G} be regular thin families with $o(\mathcal{F}) \leq o(\mathcal{G})$. Then there exists $L_0 \in [\mathbb{N}]^\infty$ such that for every $M \in [\mathbb{N}]^\infty$ there exist $N \in [\mathbb{N}]^\infty$ and $\varphi : \mathcal{G} \upharpoonright N \rightarrow \mathcal{F} \upharpoonright M$ such that $L_0(\varphi(t)) \subseteq t$, for all $t \in \mathcal{G} \upharpoonright N$.

PROOF. By Theorem 2.9 there exists $L_0 \in [\mathbb{N}]^\infty$ such that $\widehat{\mathcal{F}}(L_0) \subseteq \widehat{\mathcal{G}}$. Let $M \in [\mathbb{N}]^\infty$. Also notice that $\mathcal{F}(L_0)$ and \mathcal{G} are large in $L_0(M)$. Hence by Theorem 1.3 there exists $N \in [L_0(M)]^\infty$ such that $\mathcal{F}(L_0)$ and \mathcal{G} are very large in N . Since $\widehat{\mathcal{F}}(L_0) \subseteq \widehat{\mathcal{G}}$ we get that $\mathcal{F}(L_0) \upharpoonright N \subseteq \mathcal{G} \upharpoonright N$. To define the map $\varphi : \mathcal{G} \upharpoonright N \rightarrow \mathcal{F} \upharpoonright M$, let $t \in \mathcal{G} \upharpoonright N$. Then there exists a unique $\tilde{s} \in \mathcal{F}(L_0) \upharpoonright N$ such that $\tilde{s} \subseteq t$. We set $\varphi(t) = s$, where s is the unique element of \mathcal{F} with $L_0(s) = \tilde{s}$. \square

An immediate consequence of the above proposition is the following corollary which is related to Theorem 1.23 and Remark 1.24.

Corollary 2.12. Let \mathcal{F}, \mathcal{G} be regular thin families with $o(\mathcal{F}) \leq o(\mathcal{G})$. Then for every $M \in [\mathbb{N}]^\infty$ there exist $N \in [\mathbb{N}]^\infty$ and a plegma canonical map $\varphi : \mathcal{G} \upharpoonright N \rightarrow \mathcal{F} \upharpoonright M$. Moreover for every $l \in \mathbb{N}$ and every plegma l -tuple $(s_i)_{i=1}^l$ in $\mathcal{G} \upharpoonright N$ the l -tuple $(\varphi(s_i))_{i=1}^l$ is also a plegma in $\mathcal{F} \upharpoonright M$.

2.0.2. *Classification of \mathcal{F} -spreading models.*

Proposition 2.13. Let \mathcal{F}, \mathcal{G} be regular thin families with $o(\mathcal{F}) \leq o(\mathcal{G})$. Let X be a Banach space and $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in X which admits an \mathcal{F} -spreading model $(e_n)_{n \in \mathbb{N}}$. Then there exists a \mathcal{G} -sequence $(x'_t)_{t \in \mathcal{G}}$ such that $\{x'_t : t \in \mathcal{G}\} \subseteq \{x_s : s \in \mathcal{F}\}$, which admits the same sequence $(e_n)_{n \in \mathbb{N}}$ as a \mathcal{G} -spreading model.

PROOF. Let $M \in [\mathbb{N}]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$. By Proposition 2.11 there exist $N \in [\mathbb{N}]^\infty$ and $\varphi : \mathcal{G} \upharpoonright N \rightarrow \mathcal{F} \upharpoonright M$ such that $L_0(\varphi(t)) \subseteq t$, for all $t \in \mathcal{G} \upharpoonright N$. Using the map φ we define a \mathcal{G} -sequence $(x'_t)_{t \in \mathcal{G}}$ as follows. For every $t \in \mathcal{G} \upharpoonright N$ we set $x'_t = x_{\varphi(t)}$ and for every $t \in \mathcal{G} \setminus (\mathcal{G} \upharpoonright N)$ let x'_t be an arbitrary element of $\{x_s : s \in \mathcal{F}\}$. We will show that $(x'_t)_{t \in \mathcal{G} \upharpoonright N}$ generates $(e_n)_{n \in \mathbb{N}}$. Indeed, let $(\delta_n) \searrow 0$ such that

$$\left| \left\| \sum_{j=1}^k a_j x_{s_j} \right\| - \left\| \sum_{j=1}^k a_j e_j \right\|_* \right| \leq \delta_l$$

for every $1 \leq k \leq l$, every plegma k -tuple $(s_j)_{j=1}^k$ in $\mathcal{F} \upharpoonright M$ with $s_1(1) \geq M(l)$ and every choice of $a_1, \dots, a_k \in [-1, 1]$. Let $l \in \mathbb{N}$, $1 \leq k \leq l$, $(t_j)_{j=1}^k$ be a plegma k -tuple in $\mathcal{G} \upharpoonright N$ with $t_1(1) \geq N(l)$ and $a_1, \dots, a_k \in [-1, 1]$. Let $s_j = \varphi(t_j) \in \mathcal{F} \upharpoonright M$, for all $1 \leq j \leq k$. By Corollary 2.12, we have that $(s_j)_{j=1}^k$ is plegma. Moreover since $N \in [L_0(M)]^\infty$, $s_1(1) \geq M(l)$. Therefore,

$$\left| \left\| \sum_{j=1}^k a_j x'_{t_j} \right\| - \left\| \sum_{j=1}^k a_j e_j \right\| \right| = \left| \left\| \sum_{j=1}^k a_j x_{s_j} \right\| - \left\| \sum_{j=1}^k a_j e_j \right\|_* \right| \leq \delta_l$$

□

Corollary 2.14. Let \mathcal{F}, \mathcal{G} be regular thin families and $o(\mathcal{F}) = o(\mathcal{G})$. Then $(e_n)_{n \in \mathbb{N}}$ is an \mathcal{F} -spreading model of A iff $(e_n)_{n \in \mathbb{N}}$ is a \mathcal{G} -spreading model of A .

The above permits us to give the following definition.

Definition 2.15. Let A be a subset of a Banach space X and $1 \leq \xi < \omega_1$ be a countable ordinal. We will say that $(e_n)_{n \in \mathbb{N}}$ is a ξ -spreading model of A if there exists a regular thin family \mathcal{F} with $o(\mathcal{F}) = \xi$ such that $(e_n)_{n \in \mathbb{N}}$ is an \mathcal{F} -spreading model of A . The set of all ξ -spreading models of A will be denoted by $\mathcal{SM}_\xi(A)$.

Proposition 2.13 also yields the following.

Corollary 2.16. Let X be a Banach space and $A \subseteq X$. Then $\mathcal{SM}_\zeta(A) \subseteq \mathcal{SM}_\xi(A)$, for all $1 \leq \zeta < \xi < \omega_1$.

In the end of this section, for every $1 \leq \xi < \omega_1$ we give an example of a space X_ξ with a Schauder basis and a subset $A \subseteq X_\xi$ (actually A is consisted by the elements of the Schauder basis of X_ξ) such that there exists a sequence $(e_n)_{n \in \mathbb{N}} \in \mathcal{SM}_\xi(A)$ which is not equivalent to any $(y_n)_{n \in \mathbb{N}}$ in $\mathcal{SM}_\zeta(A)$ for all $\zeta < \xi$.

The above reveals the following problems.

Problem 1. Let $\xi < \omega_1$. Does there exist a Banach space X and $(e_n)_{n \in \mathbb{N}} \in \mathcal{SM}_\xi(X)$ such that $(e_n)_{n \in \mathbb{N}}$ is not equivalent to any $(y_n)_{n \in \mathbb{N}}$ in $\mathcal{SM}_\zeta(X)$ for all $\zeta < \xi$?

In Chapter 8, Problem 1 is answered affirmatively for all finite as well as all countable limit ordinals.

Problem 2. Does for every separable Banach space X exist $\xi < \omega_1$ such that for every $\zeta > \xi$, $\mathcal{SM}_\zeta(X) = \mathcal{SM}_\xi(X)$?

Problem 2 can be also stated in an isomorphic version, i.e. every sequence in $\mathcal{SM}_\zeta(X)$ is equivalent to some sequence in $\mathcal{SM}_\xi(X)$ and vice versa. Any version of Problem 2 remains open.

We close this section by giving an example which shows that for every $\xi < \omega_1$ there exists a Banach space X_ξ such that its base generates an ℓ^1 spreading model of order ξ while it does not admit any ℓ^1 spreading model of order ζ , for all $\zeta < \xi$.

Example 1. Let $1 \leq \xi < \omega_1$ and \mathcal{F} be a regular thin family of order ξ . We denote by $(e_s)_{s \in \mathcal{F}}$ the natural Hamel basis of $c_{00}(\mathcal{F})$. For $x \in c_{00}(\mathcal{F}_\xi)$ we set

$$\|x\| = \sup \left\{ \sum_{i=1}^l |x(s_i)| : l \in \mathbb{N}, (s_i)_{i=1}^l \in Plm_l(\mathcal{F}) \text{ and } l \leq s_1(1) \right\}$$

We set $X_\xi = \overline{(c_{00}(\mathcal{F}), \|\cdot\|)}$ and $A = \{e_s : s \in \mathcal{F}\}$. It is easy to see that the natural basis $(e_n)_{n \in \mathbb{N}}$ of ℓ^1 belongs to $\mathcal{SM}_\xi(A)$. We shall show that every $(x_n)_{n \in \mathbb{N}} \in \mathcal{SM}_\zeta(A)$ with $\zeta < \xi$ is not equivalent to the ℓ^1 basis.

Indeed, assume on the contrary that there exist a regular thin family \mathcal{G} with $o(\mathcal{G}) < \xi$ and $(x_t)_{t \in \mathcal{G}}$ a \mathcal{G} -sequence in A admitting a \mathcal{G} -spreading model $(w_n)_{n \in \mathbb{N}}$ equivalent to the ℓ^1 basis. Let $\varphi : \mathcal{G} \rightarrow \mathcal{F}$ defined by the rule $\varphi(t) = s$ if $x_t = e_s$. Let $L \in [\mathbb{N}]^\infty$ such that $(x_t)_{t \in \mathcal{G} \upharpoonright L}$ generates as \mathcal{G} -spreading model the sequence $(w_n)_{n \in \mathbb{N}}$. Theorem 1.23 yields that there exists $M \in [L]^\infty$ such that for every plegma pair (t_1, t_2) in $\mathcal{G} \upharpoonright M$ neither $(\varphi(t_1), \varphi(t_2))$, nor $(\varphi(t_2), \varphi(t_1))$ is a plegma pair. Since $(x_t)_{t \in \mathcal{G} \upharpoonright M}$ also generates $(w_n)_{n \in \mathbb{N}}$ as a \mathcal{G} -spreading model there exists $c > 0$ such that for every $k \in \mathbb{N}$ there exists $(t_1, \dots, t_k) \in Plm_k(\mathcal{G} \upharpoonright M)$ with $(x_{t_1}, \dots, x_{t_k})$ c -equivalent to the natural basis (e_1, \dots, e_k) of ℓ_k^1 . Since $(\varphi(t_1), \dots, \varphi(t_k))$ cannot contain a plegma pair, by the definition of the norm, we have that $(e_{\varphi(t_1)}, \dots, e_{\varphi(t_k)})$ is isometric to the basis of ℓ_k^∞ . Since $e_{\varphi(t_i)} = x_{t_i}$ we conclude that there exists $c > 0$ such that for every k , ℓ_k^∞ is c equivalent to ℓ_k^1 and this yields a contradiction.

CHAPTER 3

Topological and combinatorial features of \mathcal{F} -sequences and norm properties of \mathcal{F} -spreading models

In this chapter we present several topological and Ramsey results concerning \mathcal{F} -sequences for an arbitrary regular thin family \mathcal{F} . Our main aim is to apply them in the theory of \mathcal{F} spreading models and in particular to provide necessary and sufficient conditions for an \mathcal{F} -spreading model to be either a nontrivial or a Schauder basic or an unconditional one.

1. \mathcal{F} -sequences in topological spaces

In this section we deal with the topological properties of \mathcal{F} -sequences in topological spaces (X, \mathcal{T}) . In the first two subsections we extend the classical definition of convergent and Cauchy sequences and we present some related results. In the third subsection we proceed to the study of \mathcal{F} -sequences with relatively compact metrizable range. Specifically, the concept of the *subordinated* \mathcal{F} -sequence is introduced in order to capture the following topological fact: Every map $\varphi : \mathcal{F} \rightarrow (X, \mathcal{T})$, with $\overline{\varphi[\mathcal{F}]}$ compact metrizable, is *locally continuously extendable* to all of $\widehat{\mathcal{F}}$, i.e. for every $M \in [\mathbb{N}]^\infty$ there exist $L \in [M]^\infty$ and a continuous extension $\widehat{\varphi} : \widehat{\mathcal{F}} \upharpoonright L \rightarrow (X, \mathcal{T})$.

1.1. Convergence of \mathcal{F} -sequences.

Definition 3.1. Let (X, \mathcal{T}) be a topological space, \mathcal{F} a regular thin family, $M \in [\mathbb{N}]^\infty$, $x_0 \in X$ and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in X . We will say that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ converges to x_0 if for every open neighborhood U of x_0 there exists $m \in \mathbb{N}$ such that for every $s \in \mathcal{F} \upharpoonright M$ with $\min s \geq M(m)$ we have that $x_s \in U$.

Remark 3.2.

- (i) It is immediate that if an \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ in a topological space is convergent to some x_0 , then every further \mathcal{F} -subsequence is also convergent to x_0 .
- (ii) It is also easy to see that for families \mathcal{F} with $o(\mathcal{F}) \geq 2$, the convergence of an \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ does not in general imply the corresponding relative compactness of its range.
- (iii) Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence to some x_0 in a topological space. Then for every regular thin family \mathcal{F} and every 1-1 map from \mathcal{F} to $\{x_n\}_{n \in \mathbb{N}}$ the corresponding \mathcal{F} -sequence is convergent to x_0 .

Definition 3.3. Let (X, ρ) be a metric space, \mathcal{F} a regular thin family, $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in Y . We will say that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is Cauchy if for every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that for every $s_1, s_2 \in \mathcal{F} \upharpoonright M$ with $\min s_1, \min s_2 \geq M(m)$, we have that $\rho(x_{s_1}, x_{s_2}) < \varepsilon$.

Proposition 3.4. Let $M \in [\mathbb{N}]^\infty$, \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in a complete metric space (X, ρ) . Then the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is Cauchy if and only if $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is convergent.

PROOF. If the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is convergent, then it is straightforward that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is Cauchy. Concerning the converse we have the following. Suppose that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is Cauchy. Let $(s_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{F} \upharpoonright M$ such that $\min s_n \rightarrow \infty$. It is immediate that $(x_{s_n})_{n \in \mathbb{N}}$ forms a Cauchy sequence in X . Since (X, ρ) is complete, there exists $x \in X$ such that the sequence $(x_{s_n})_{n \in \mathbb{N}}$ converges to x . We will show that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ converges to x . Indeed, let $\varepsilon > 0$. Since $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is Cauchy, there exists $k_0 \in \mathbb{N}$ such that for every $t_1, t_2 \in \mathcal{F} \upharpoonright M$ with $\min t_1, \min t_2 \geq M(k_0)$ we have that $\rho(x_{t_1}, x_{t_2}) < \frac{\varepsilon}{2}$. Since the sequence $(x_{s_n})_{n \in \mathbb{N}}$ converges to x and $\min s_n \rightarrow \infty$, there exists $n_0 \in \mathbb{N}$ such that $\min s_{n_0} \geq M(k_0)$ and $\rho(x, x_{s_{n_0}}) < \varepsilon/2$. Hence for every $s \in \mathcal{F} \upharpoonright M$ such that $\min s \geq M(k_0)$, we have that $\rho(x, x_s) \leq \rho(x, x_{s_{n_0}}) + \rho(x_{s_{n_0}}, x_s) < \varepsilon$ and the proof is completed. \square

1.2. Plegma ε -separated \mathcal{F} -sequences.

Lemma 3.5. Let $M \in [\mathbb{N}]^\infty$, \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in a metric space (X, ρ) . Suppose that for every $\varepsilon > 0$ and every $L \in [M]^\infty$ there exists a plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L$ such that $\rho(x_{s_1}, x_{s_2}) < \varepsilon$. Then the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ has a further Cauchy subsequence.

PROOF. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive reals such that $\sum_{n=1}^\infty \varepsilon_n < \infty$. Using Proposition 1.12, we inductively construct a decreasing sequence $(L_n)_{n \in \mathbb{N}}$ in $[M]^\infty$, such that for every $n \in \mathbb{N}$ and for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L_n$ we have that $\rho(x_{s_1}, x_{s_2}) < \varepsilon_n$. Let L' be a diagonalization of $(L_n)_{n \in \mathbb{N}}$, i.e. $L'(n) \in L_n$ for all $n \in \mathbb{N}$, and $L = \{L'(2n) : n \in \mathbb{N}\}$.

We claim that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is Cauchy. Indeed let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^\infty \varepsilon_n < \frac{\varepsilon}{2}$. Let s_0 be the unique initial segment of $\{L(n) : n \geq n_0\}$ in \mathcal{F} . If $\max s_0 = L(k)$ then we set $k_0 = k + 1$. Then for every $s_1, s_2 \in \mathcal{F} \upharpoonright L$ with $\min s_1, \min s_2 \geq L(k_0)$, by Proposition 1.17 there exist plegma paths $(s_j^1)_{j=1}^{|s_0|}, (s_j^2)_{j=1}^{|s_0|}$ in $\mathcal{F} \upharpoonright L'$ from s_0 to s_1, s_2 respectively. Then for $i = 1, 2$ we have that

$$\rho(x_{s_0}, x_{s_i}) \leq \sum_{j=0}^{|s_0|-1} \rho(x_{s_j^i}, x_{s_{j+1}^i}) < \sum_{j=0}^{|s_0|-1} \varepsilon_{n_0+j} < \frac{\varepsilon}{2}$$

which implies that $\rho(x_{s_1}, x_{s_2}) < \varepsilon$. \square

Definition 3.6. Let $\varepsilon > 0$, $L \in [\mathbb{N}]^\infty$, \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in a metric space X . We will say that the subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is plegma ε -separated if for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L$, $\rho(x_{s_1}, x_{s_2}) > \varepsilon$.

The following proposition is actually a restatement of Lemma 3.5 under the above definition.

Proposition 3.7. Let $M \in [\mathbb{N}]^\infty$, \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in a metric space X . Then the following are equivalent.

- (i) The \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ has no further Cauchy subsequence.
- (ii) For every $N \in [M]^\infty$ there exists $L \in [N]^\infty$ and $\varepsilon > 0$ such that the subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is plegma ε -separated.

1.3. Subordinated \mathcal{F} -sequences.

Definition 3.8. Let (X, \mathcal{T}) be a topological space, \mathcal{F} be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in X . We say that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is subordinated (with respect to (X, \mathcal{T})) if there exists a continuous map $\widehat{\varphi} : \widehat{\mathcal{F}} \upharpoonright M \rightarrow (X, \mathcal{T})$ with $\widehat{\varphi}(s) = x_s$, for all $s \in \mathcal{F} \upharpoonright M$.

Remark 3.9. If $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is subordinated then we have that

$$\overline{\{x_s : s \in \mathcal{F} \upharpoonright M\}} = \widehat{\varphi}[\widehat{\mathcal{F}} \upharpoonright M],$$

where $\overline{\{x_s : s \in \mathcal{F} \upharpoonright M\}}$ is the \mathcal{T} -closure of $\{x_s : s \in \mathcal{F} \upharpoonright M\}$ in X . Therefore $\overline{\{x_s : s \in \mathcal{F} \upharpoonright M\}}$ is a countable compact metrizable subspace of (X, \mathcal{T}) with Cantor-Bendixson index at most $o(\mathcal{F}) + 1$. Moreover notice that if $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is subordinated then for every $L \in [M]^\infty$, we have that $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is subordinated.

Proposition 3.10. Let (X, \mathcal{T}) be a Hausdorff topological space, \mathcal{F} a regular thin family, $M \in [\mathbb{N}]^\infty$, $x_0 \in X$ and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in X . Suppose that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is convergent to x_0 and subordinated. Let $\widehat{\varphi} : \widehat{\mathcal{F}} \upharpoonright M \rightarrow (X, \mathcal{T})$ be the continuous map witnessing it. Then $\widehat{\varphi}(\emptyset) = x_0$.

PROOF. By passing to a further infinite subset of M if it is necessary, we may assume that \mathcal{F} is very large in M . Assume on the contrary that $x_0 \neq \widehat{\varphi}(\emptyset)$. Since (X, \mathcal{T}) is Hausdorff there exist $V, V' \in \mathcal{T}$ with $x_0 \in V$ and $\widehat{\varphi}(\emptyset) \in V'$ such that $V \cap V' = \emptyset$. By the convergence of $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ to x_0 , there exists $m \in \mathbb{N}$ such that $x_s \in V$ for every $s \in \mathcal{F} \upharpoonright M$ with $\min s \geq M(m)$. By the continuity of $\widehat{\varphi}$ there exists $m_1 \in M$ with $m_1 \geq M(m)$ and $\widehat{\varphi}(\{m_1\}) \in V'$. Similarly there exists $m_2 \in M$ with $m_2 > m_1$ such that $\widehat{\varphi}(\{m_1, m_2\}) \in V'$ and so on. Since \mathcal{F} is very large in M , there exists a $k \in \mathbb{N}$ such that $s = \{m_1, \dots, m_k\} \in \mathcal{F}$ and $\widehat{\varphi}(s) = x_s \in V'$. Moreover, since $\min(s) \geq M(m)$, we should also have $x_s \in V$, which is a contradiction. \square

Proposition 3.11. Let (X, \mathcal{T}) be a topological space, \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in X such that $\overline{\{x_s : s \in \mathcal{F}\}}$ is a compact metrizable subspace of (X, \mathcal{T}) . Then for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is subordinated.

PROOF. We will use induction on the order of \mathcal{F} . If $o(\mathcal{F}) = 0$, i.e $\mathcal{F} = \{\emptyset\}$ there is nothing to prove. Assume that for some $\xi < \omega_1$, the conclusion of the proposition holds for every family of order lower than ξ . Let \mathcal{F} be a regular thin family of order ξ and $M \in [\mathbb{N}]^\infty$. For every $n \in M$ we have that $o(\mathcal{F}_{(n)}) < o(\mathcal{F}) = \xi$. Let $L_0 = M$ and $l_1 = \min L_0$. For every $s \in \mathcal{F}_{(l_1)}$, we set $x_s^1 = x_{\{l_1\} \cup s}$. Since $\mathcal{F}_{(l_1)}$ is also a regular thin family (see Remark 1.4), by the inductive hypothesis there exists $L_1 \in [L_0 \setminus \{l_1\}]^\infty$ such that $(x_s^1)_{s \in \mathcal{F}_{(l_1)} \upharpoonright L_1}$ is subordinated. Moreover since $\overline{\{x_s^1 : s \in \mathcal{F}_{(l_1)} \upharpoonright L_1\}}$ is compact metrizable, using Theorem 1.3 and Remark 3.9 we may also suppose that $\text{diam}(\{x_s^1 : s \in \mathcal{F}_{(l_1)} \upharpoonright L_1\}) < 1$. Proceeding in the same way, we construct by induction a strictly increasing sequence $(l_n)_{n \in \mathbb{N}}$, a decreasing sequence $(L_n)_{n=0}^\infty$ of infinite subsets of \mathbb{N} , such that setting $x_s^n = x_{\{l_n\} \cup s}$ for all $s \in \mathcal{F}_{(l_n)} \upharpoonright L_n$ the following are satisfied

- (i) For every $n \in \mathbb{N}$, $l_n = \min L_{n-1}$ and $L_n \in [L_{n-1} \setminus \{l_n\}]^\infty$.
- (ii) The sequence $(x_s^n)_{s \in \mathcal{F}_{(l_n)} \upharpoonright L_n}$ is subordinated.
- (iii) $\text{diam}(\{x_s^n : s \in \mathcal{F}_{(l_n)} \upharpoonright L_n\}) < \frac{1}{n}$.

Let $\widehat{\varphi}_n$ be the map witnessing that $(x_s)_{s \in \mathcal{F}(l_n)}$ is subordinated. Let $L' = \{l_n : n \in \mathbb{N}\}$. Let $x_n = \widehat{\varphi}_n(\emptyset)$, $n \in \mathbb{N}$ and choose $N \in [\mathbb{N}]^\infty$ such that the sequence $(x_n)_{n \in N}$ is convergent. We set $L = L'(N)$ and we define $\widehat{\varphi} : \widehat{\mathcal{F} \upharpoonright L} \rightarrow X$ as follows. For $s = \emptyset$, we set $\widehat{\varphi}(\emptyset) = \lim_{n \in N} x_n$ and for $s \neq \emptyset$, we set $\widehat{\varphi}(s) = \widehat{\varphi}_n(s \setminus \{l_n\})$, where $l_n = \min s$. Using that $\lim_{n \rightarrow \infty} \text{diam}\{\widehat{\varphi}(s) : s \in \mathcal{F} \upharpoonright L \text{ and } \min s = L(n)\} \rightarrow 0$ and hence $\widehat{\varphi}$ is continuous at $s = \emptyset$. For all other $s \in \widehat{\mathcal{F} \upharpoonright L}$ the continuity of $\widehat{\varphi}$ follows easily by the continuity of $\widehat{\varphi}_n$. \square

2. Trivial spreading models

In this section we provide equivalent conditions for the seminorm $\|\cdot\|_*$ of a spreading model of any order to be a norm. To this end we define the trivial sequences in a seminormed space and we show that the seminorm $\|\cdot\|_*$ is not a norm iff the spreading model is trivial. Moreover we study the topological behavior of \mathcal{F} -sequences which generate trivial spreading models. Among others we prove that an \mathcal{F} -sequence admits a trivial spreading model iff it contains a norm Cauchy subsequence. This generalizes the classical Brunel-Sucheston condition for spreading models of any order.

Proposition 3.12. Let $(E, \|\cdot\|_*)$ be an infinite dimensional seminormed linear space with Hamel basis $(e_n)_{n \in \mathbb{N}}$ such that $(e_n)_{n \in \mathbb{N}}$ is 1-subsymmetric with respect to $\|\cdot\|_*$. Then the following are equivalent:

- (i) The seminorm $\|\cdot\|_*$ on E is not a norm, i.e. there exists $x \in E$ such that $x \neq 0$ and $\|x\|_* = 0$.
- (ii) For every $n, m \in \mathbb{N}$, $\|e_n - e_m\|_* = 0$.
- (iii) For every $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$, we have that

$$\left\| \sum_{i=1}^n a_i e_i \right\|_* = \left| \sum_{i=1}^n a_i \right| \cdot \|e_1\|_*$$

PROOF. The implication (ii) \Rightarrow (i) is straightforward. To show the converse let $x = \sum_{j=1}^n a_j e_j \in E$ such that $x \neq 0$ and $\|x\|_* = 0$. By the linear independence and 1-subsymmetricity of $(e_n)_{n \in \mathbb{N}}$ we may suppose that $a_j \neq 0$ for all $1 \leq j \leq n$. Again by the 1-subsymmetricity of $(e_n)_{n \in \mathbb{N}}$ we have that

$$\left\| \sum_{j=1}^{n-1} a_j e_j + a_n e_n \right\|_* = \left\| \sum_{j=1}^{n-1} a_j e_j + a_n e_{n+1} \right\|_* = 0$$

which yields that

$$\|e_n - e_{n+1}\|_* \leq \frac{1}{|a_n|} \left(\left\| \sum_{j=1}^{n-1} a_j e_j + a_n e_n \right\|_* + \left\| \sum_{j=1}^{n-1} a_j e_j + a_n e_{n+1} \right\|_* \right) = 0$$

By the 1-subsymmetricity of $(e_n)_{n \in \mathbb{N}}$ the result follows.

The equivalence between (ii) and (iii) is immediate using the 1-subsymmetricity of the sequence $(e_n)_{n \in \mathbb{N}}$. \square

Definition 3.13. Let $(E, \|\cdot\|_*)$ be a seminormed space and $(e_n)_{n \in \mathbb{N}}$ be a sequence in E . Then $(e_n)_{n \in \mathbb{N}}$ will be called trivial if

$$(4) \quad \left\| \sum_{i=1}^n a_i e_i \right\|_* = \left| \sum_{i=1}^n a_i \right| \cdot \|e_1\|_*$$

for all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$.

Remark 3.14. We note that (4) in the above definition yields that a sequence $(e_n)_{n \in \mathbb{N}}$ is not a Schauder basic sequence. In particular if we assume that the associated projections $\sum_{i=1}^m a_i e_i \xrightarrow{P_q} \sum_{i=1}^n a_i e_i$ are uniformly bounded then $\|e_n\|_* = 0$ for all $n \in \mathbb{N}$.

Remark 3.15. First notice that every trivial sequence $(e_n)_{n \in \mathbb{N}}$ in any seminormed space $(E, \|\cdot\|_*)$ is 1-subsymmetric. Moreover we may point out the following concerning trivial sequences.

Let \mathcal{F} be a regular thin family. Then every constant \mathcal{F} -sequence $(x_s)_{s \in \mathcal{F}}$ in any Banach space X , i.e. there exists $x \in X$ such that $x_s = x$ for all $s \in \mathcal{F}$, generates a trivial \mathcal{F} -spreading model. Conversely if $(e_n)_{n \in \mathbb{N}}$ is a trivial Hamel basis of a seminormed space $(E, \|\cdot\|_*)$, then $(e_n)_{n \in \mathbb{N}}$ can be generated as an \mathcal{F} -spreading model of any constant \mathcal{F} -sequence $(x_s)_{s \in \mathcal{F}}$ in a Banach space X with $x_s = x \in X$ for all $s \in \mathcal{F}$ and $\|x\| = \|e_1\|_*$.

Theorem 3.16. Let $M \in [\mathbb{N}]^\infty$, \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in a Banach space X . Let $(E, \|\cdot\|_*)$ be an infinite dimensional seminormed linear space with Hamel basis $(e_n)_{n \in \mathbb{N}}$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model. Then the following are equivalent:

- (i) The seminorm $\|\cdot\|_*$ is not a norm on E .
- (ii) The sequence $(e_n)_{n \in \mathbb{N}}$ is trivial.
- (iii) The sequence $(e_n)_{n \in \mathbb{N}}$ is generated as a \mathcal{G} -spreading model by any constant \mathcal{G} -sequence $(y_t)_{t \in \mathcal{G}}$ in any Banach space Y , such that $\|y_t\| = \|e_1\|_*$ for all $t \in \mathcal{G}$ and for every regular thin family \mathcal{G} .
- (iv) The \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ contains a further norm Cauchy subsequence.
- (v) For every $\varepsilon > 0$ and every $L \in [M]^\infty$, the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is not plegma ε -separated.
- (vi) There exists $x \in X$ such that every subsequence of $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ contains a further subsequence convergent to x .

PROOF. The equivalence of (i), (ii), (iii) follows by Proposition 3.12 and Remark 3.15.

(ii) \Rightarrow (v): Let $\varepsilon > 0$ and $L \in [M]^\infty$. Since the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ also generates the trivial sequence $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model, there exists $n_0 \in \mathbb{N}$ such that for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L$ with $\min s_1, \min s_2 \geq L(n_0)$, we have that

$$\left\| x_{s_1} - x_{s_2} \right\| = \left| \left\| x_{s_1} - x_{s_2} \right\| - \left\| e_1 - e_2 \right\|_* \right| < \varepsilon$$

and therefore the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is not plegma ε -separated.

(v) \Rightarrow (iv): This follows by Lemma 3.5.

(iv) \Rightarrow (ii): We can easily construct a sequence $((s_1^n, s_2^n))_{n \in \mathbb{N}}$ of plegma pairs in $\mathcal{F} \upharpoonright M$ such that $s_1^n(1) \rightarrow \infty$ and $\|x_{s_1^n} - x_{s_2^n}\| < \frac{1}{n}$. Hence

$$\|e_1 - e_2\|_* = \lim_{n \rightarrow \infty} \|x_{s_1^n} - x_{s_2^n}\| = 0$$

Thus by Proposition 3.12 we get (ii). By the above we have that (i)-(v) are equivalent.

It remains to show that (ii) and (vi) are equivalent. It is straightforward that (vi) \Rightarrow (iv) \Rightarrow (ii). Conversely suppose that (ii) holds. Then every subsequence of $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model. Hence, by the equivalence of

(ii) and (iv) and Proposition 3.4, every subsequence of $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ contains a further convergent subsequence. It remains to show that all the convergent subsequences of $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ have a common limit. Let $L_1, L_2 \in [M]^\infty$ and $x_1, x_2 \in X$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright L_1}$ converges to x_1 and $(x_s)_{s \in \mathcal{F} \upharpoonright L_2}$ converges to x_2 . We will show that $x_1 = x_2$. Let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for every $s_1 \in \mathcal{F} \upharpoonright L_1$ with $\min s_1 \geq L_1(n_0)$ and $s_2 \in \mathcal{F} \upharpoonright L_2$ with $\min s_2 \geq L_2(n_0)$ we have that $\|x_1 - x_{s_1}\| < \frac{\varepsilon}{3}$ and $\|x_2 - x_{s_2}\| < \frac{\varepsilon}{3}$. Notice that n_0 can be chosen large enough such that for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright M$ with $\min s_1 \geq M(n_0)$,

$$\|x_{s_1} - x_{s_2}\| = \left| \|x_{s_1} - x_{s_2}\| - \|e_1 - e_2\|_* \right| < \frac{\varepsilon}{3}$$

It is easy to see that we can choose $s_1 \in \mathcal{F} \upharpoonright L_1$ with $\min s_1 \geq L_1(n_0)$ and $s_2 \in \mathcal{F} \upharpoonright L_2$ with $\min s_2 \geq L_2(n_0)$ such that the pair (s_1, s_2) is plegma. Then

$$\|x_1 - x_2\| \leq \|x_1 - x_{s_1}\| + \|x_{s_1} - x_{s_2}\| + \|x_2 - x_{s_2}\| < \varepsilon$$

It follows that $x_1 = x_2$ and the proof is complete. \square

Remark 3.17. Let us notice that a 1-subsymmetric linearly independent sequence in a Banach space is not necessarily a Schauder basic sequence. Indeed, consider the sequence $(x_n)_{n \in \mathbb{N}}$ in $\ell^2(\mathbb{N})$, with $x_n = e_1 + e_{n+1}$ for all $n \in \mathbb{N}$, where $(e_n)_{n \in \mathbb{N}}$ is the usual basis of ℓ^2 . It is immediate that $(x_n)_{n \in \mathbb{N}}$ is linearly independent and 1-subsymmetric. Since $(x_n)_{n \in \mathbb{N}}$ is weakly convergent to $e_1 \neq 0$, we have that $(x_n)_{n \in \mathbb{N}}$ is not Schauder basic.

3. Ramsey properties of \mathcal{F} -sequences and applications to Schauder basic spreading models

Our main aim in this section is to give sufficient conditions for a spreading model of any order to be a Schauder basic sequence. This requires to develop some Ramsey theoretic machinery concerning maps on regular thin families and this is the content of the first two subsections. Specifically, in the first subsection we present some elements of abstract Ramsey theory which will be used in the second one, to obtain some combinatorial properties concerning \mathcal{F} -sequences. In the last subsection we apply these combinatorial results and we extend the well known fact that every spreading model admitted by a seminormalized Schauder basic sequence is also Schauder basic.

3.1. Elements of Abstract Ramsey Theory. In this subsection we review some well known Abstract Ramsey properties which concern families of pairs of the form $(t, Y) \in [\mathbb{N}]^{<\infty} \times [\mathbb{N}]^\infty$. Our main aim is to prove Proposition 3.21 below which will be used in the next subsection. Let us point out that a combination of Theorem 2 in [25] and results from [20] would also yield Proposition 3.21. As the full strength of Theorem 2 in [25] is actually not necessary for the results included here, we have chosen to present a self contained and direct proof.

Lemma 3.18. Let $X \in [\mathbb{N}]^\infty$ and $\mathcal{Q} \subseteq [X]^{<\infty} \times [X]^\infty$ which satisfies the following properties.

- (P1) (Hereditariness) For every $(t, Y) \in \mathcal{Q}$ and $Z \in [Y]^\infty$ we have that $(t, Z) \in \mathcal{Q}$.
- (P2) (Cofinality) For every $(t, Y) \in [X]^{<\infty} \times [X]^\infty$ there exists $Z \in [Y]^\infty$ such that $(t, Z) \in \mathcal{Q}$.

(P3) For every $(t, Y) \in \mathcal{Q}$ and $Z \in [X]^\infty$ with $Y \cap (\max t, \infty) = Z \cap (\max t, \infty)$, we have that $(t, Z) \in \mathcal{Q}$.

Then for every $Y \in [X]^\infty$ there exists $Z \in [Y]^\infty$ such that for every $t \in [Z]^{<\infty}$, we have that $(t, Z) \in \mathcal{Q}$.

Let $N \in [\mathbb{N}]^\infty$ and $\mathcal{A} \subseteq [N]^{<\infty} \times [N]^\infty$. For a pair $(t, X) \in [N]^{<\infty} \times [N]^\infty$ we will say that

- (i) X *accepts* t if $(t, X) \in \mathcal{A}$.
- (ii) X *rejects* t if for every $Y \in [X]^\infty$, Y does not accept t .
- (iii) X *decides* t if either X accepts t , or X rejects t .

Remark 3.19. Given $N \in [\mathbb{N}]^\infty$ and $\mathcal{A} \subseteq [N]^{<\infty} \times [N]^\infty$, let \mathcal{R} (resp. \mathcal{D}) to be the set of all pairs (t, X) in $[N]^{<\infty} \times [N]^\infty$ such that X rejects t (resp. X decides t). It is easy to check the following:

- (i) The family \mathcal{R} is hereditary and the family \mathcal{D} is cofinal.
- (ii) If \mathcal{A} is hereditary, then \mathcal{D} is hereditary and \mathcal{R} satisfies (P3).
- (iii) Suppose that properties (P1) and (P3) of Lemma 3.18 are satisfied for $\mathcal{Q} = \mathcal{A}$ and $X = N$. Then (P1)-(P3) are also satisfied for $\mathcal{Q} = \mathcal{D}$ and $X = N$. Hence by Lemma 3.18 we get $M \in [N]^\infty$ such that $[M]^{<\infty} \times [M]^\infty \subseteq \mathcal{D}$.

Given $N \in [\mathbb{N}]^\infty$ and $\mathcal{A} \subseteq [N]^{<\infty} \times [N]^\infty$, for a pair $(t, X) \in [N]^{<\infty} \times [N]^\infty$ we will say that X *uniformly decides* $t \cup \{n\}$, for all $n \in X$, if either for every $n \in X$, X accepts $t \cup \{n\}$, or for every $n \in X$, X rejects $t \cup \{n\}$.

Remark 3.20. Let $N \in [\mathbb{N}]^\infty$ and $\mathcal{A} \subseteq [N]^{<\infty} \times [N]^\infty$. Suppose that properties (P1) and (P3) of Lemma 3.18 are satisfied for $\mathcal{Q} = \mathcal{A}$ and $X = N$. Let \mathcal{R} and \mathcal{D} as in Remark 3.19. Then by part (iii) of the same remark there exists $M \in [N]^\infty$ such that $[M]^{<\infty} \times [M]^\infty \subseteq \mathcal{D}$. Let

$$\mathcal{D}' = \left\{ (t, X) \in [M]^{<\infty} \times [M]^\infty : X \text{ uniformly decides } t \cup \{n\}, \text{ for all } n \in X \right\}$$

It is easy to see that properties (P1)-(P3) of Lemma 3.18 are satisfied for $X = M$ and $\mathcal{Q} = \mathcal{D}'$. Indeed, (P1) and (P3) are immediate and (P2) follows by the pigeonhole principle.

The above remark yields easily the following.

Proposition 3.21. Let $N \in [\mathbb{N}]^\infty$ and $\mathcal{A} \subseteq [N]^{<\infty} \times [N]^\infty$ such that properties (P1) and (P3) of Lemma 3.18 are satisfied for $\mathcal{Q} = \mathcal{A}$ and $X = N$. Then there exists $L \in [N]^\infty$ such that for every $t \in [L]^{<\infty}$ the following are satisfied.

- (i) L decides t and
- (ii) L uniformly decides $t \cup \{n\}$, for all $n \in L$ with $n > \max t$.

3.2. Combinatorial properties of \mathcal{F} -sequences.

Definition 3.22. Let A be a set, $M \in [\mathbb{N}]^\infty$, $\mathcal{F} \subseteq [M]^{<\infty}$ and $\varphi : \mathcal{F} \rightarrow A$. We will say that φ is *hereditarily nonconstant* in M if for every $L \in [M]^\infty$ the restriction of φ on $\mathcal{F} \upharpoonright L$ is nonconstant. In particular if $M = \mathbb{N}$ then we will say that φ is hereditarily nonconstant.

Proposition 3.23. Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ be a regular thin family, A be a set and $\varphi : \mathcal{F} \rightarrow A$ be hereditarily nonconstant. Then for every $N \in [\mathbb{N}]^\infty$ there exists $L \in [N]^\infty$ such that for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L$, $\varphi(s_1) \neq \varphi(s_2)$.

PROOF. By Proposition 1.12 there exists an $L \in [N]^\infty$ such that either $\varphi(s_1) \neq \varphi(s_2)$, for all plegma pairs (s_1, s_2) in $\mathcal{F} \upharpoonright L$, or $\varphi(s_1) = \varphi(s_2)$, for all plegma pairs (s_1, s_2) in $\mathcal{F} \upharpoonright L$. The second alternative is excluded.

Indeed, suppose that $\varphi(s_1) = \varphi(s_2)$, for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L$. We may also assume that $\mathcal{F} \upharpoonright L$ is very large in L . Let s_0 be the unique initial segment of $L_0 = \{L(2\rho) : \rho \in \mathbb{N}\}$ in $\mathcal{F} \upharpoonright L$ and let $k = |s_0|$. We set $L'_0 = \{L(2\rho) : \rho \in \mathbb{N} \text{ and } \rho > k\}$. By Proposition 1.17 for every $s \in \mathcal{F} \upharpoonright L'_0$ there exist a plegma path $(s_0, s_1, \dots, s_{k-1}, s)$ of length k in $\mathcal{F} \upharpoonright L$. Then for every $s \in \mathcal{F} \upharpoonright L'_0$ we have that $\varphi(s) = \varphi(s_{k-1}) = \dots = \varphi(s_1) = \varphi(s_0)$, which contradicts that φ is hereditarily nonconstant. \square

For every $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and $t \in [\mathbb{N}]^{<\infty}$, we set

$$\mathcal{F}_{[t]} = \{s \in \mathcal{F} : t \sqsubseteq s\}$$

Lemma 3.24. Let $M \in [\mathbb{N}]^\infty$ and $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ thin and large in M . Let $\varphi : \mathcal{F} \rightarrow A$ be hereditarily nonconstant in M . Then there exist $L \in [M]^\infty$ and a family $\mathcal{F}_{\varphi,L} \subseteq [L]^{<\infty}$ satisfying the following properties:

- (i) $\mathcal{F}_{\varphi,L}$ is thin, very large in L with $o(\mathcal{F}_{\varphi,L}) \geq 1$ and $\mathcal{F}_{\varphi,L} \subseteq \mathcal{F} \upharpoonright L$.
- (ii) For every $t \in \mathcal{F}_{\varphi,L}$ there exists $c_t \in A$ such that $\varphi[\mathcal{F}_{[t]} \upharpoonright L] = \{c_t\}$, that is φ is constant on $\mathcal{F}_{[t]} \upharpoonright L$.
- (iii) For every $u \in \widehat{\mathcal{F}}_{\varphi,L}$ with $o_{\widehat{\mathcal{F}}_{\varphi,L}}(u) = 1$ and every $l_1 < l_2$ in L with $\max u < l_1 < l_2$ in L , the sets $t_1 = u \cup \{l_1\}, t_2 = u \cup \{l_2\}$ belong to $\mathcal{F}_{\varphi,L}$ and moreover $c_{t_1} \neq c_{t_2}$.

PROOF. First we choose $N \in [M]^\infty$ such that \mathcal{F} is very large in N . We define

$$\mathcal{A} = \left\{ (t, X) \in [N]^{<\infty} \times [N]^\infty : \text{either } \varphi[\mathcal{F}_{[t]} \upharpoonright (t \cup X)] = \emptyset \text{ or } |\varphi[\mathcal{F}_{[t]} \upharpoonright (t \cup X)]| = 1 \right\}$$

It is easy to check that \mathcal{A} satisfies the assertions of Proposition 3.21 and therefore there exists $N_1 \in [N]^\infty$ such that for every $t \in [N_1]^{<\infty}$, N_1 decides t and also uniformly decides $t \cup \{n\}$, for all $n \in N_1$ with $\max t < n$. Let

$$\mathcal{G} = \{t \in [N_1]^{<\infty} : (t, N_1) \in \mathcal{A}\}$$

and \mathcal{G}_{\min} be the set of all \sqsubseteq -minimal elements in \mathcal{G} . Notice that $\mathcal{F} \upharpoonright N_1 \subseteq \mathcal{G}$. Hence, since \mathcal{F} is very large in N_1 , we get that $\mathcal{G}_{\min} \subseteq \mathcal{F} \upharpoonright N_1$. Moreover, since φ is hereditarily nonconstant we have that N_1 rejects \emptyset . Hence $\emptyset \notin \mathcal{G}$ and $o(\mathcal{G}_{\min}) > 1$.

For every $L \in [N_1]^\infty$, we set

$$\mathcal{F}_{\varphi,L} = \mathcal{G}_{\min} \upharpoonright L$$

Then it is easy to see that for every $L \in [N_1]^\infty$, assertions (i) and (ii) of the lemma are satisfied by $\mathcal{F}_{\varphi,L}$.

It remains to determine an $L \in [N_1]^\infty$ which in addition satisfies (iii). Let $t \in \mathcal{G}_{\min}$ and $u = t \setminus \{\max t\}$, i.e. $u \in \widehat{\mathcal{G}}_{\min}$ with $o_{\widehat{\mathcal{G}}_{\min}}(u) = 1$. Since N_1 uniformly decides $u \cup \{n\}$ for all $n \in N_1$ with $n > \max u$, we conclude that $u \cup \{n\} \in \mathcal{G}_{\min}$ for all $n \in N_1$ with $n > \max u$. Moreover N_1 rejects u which implies that for every $Y \in [N_1]^\infty$ the set $\varphi[\mathcal{F}_{[u]} \upharpoonright (u \cup Y)]$ is infinite. Indeed, suppose on the contrary that there exists $Y \in [N_1]^\infty$ such that the set $\varphi[\mathcal{F}_{[u]} \upharpoonright (u \cup Y)]$ is finite. Then applying Theorem 1.5 for the thin family $\mathcal{F}_{(u)}$ and for $Y \in [N]^\infty$ we get $Z \in [Y]^\infty$ such that $\varphi[\mathcal{F}_{[u]} \upharpoonright u \cup Z]$ is a singleton, that is Z accepts u which is a contradiction.

Using the above remarks, we can easily construct by induction an increasing sequence $(l_n)_{n \in \mathbb{N}}$ in N_1 and a decreasing sequence $(L_n)_{n \in \mathbb{N}}$ in $[N_1]^\infty$ such that for every $n \in \mathbb{N}$ the following are satisfied:

- (a) $l_n = \min L_{n-1}$, where $L_0 = N_1$.
- (b) For every $u \in \widehat{\mathcal{G}_{\min}}$ with $o_{\widehat{\mathcal{G}_{\min}}}(u) = 1$, $u \subseteq \{l_1, \dots, l_n\}$ and for every $m_1 < m_2$ in L_n , we have that (i) $t_1 = u \cup \{m_1\}$, $t_2 = u \cup \{m_2\}$ belong to \mathcal{G}_{\min} and (ii) $c_{t_1} \neq c_{t_2}$.

We set $L = \{l_n : n \in \mathbb{N}\}$. It can be readily verified that L and $\mathcal{F}_{\varphi, L}$ are as desired and the proof is complete. \square

The above lemma is applied for $A = \mathbb{N}$ in the next proposition.

Proposition 3.25. Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $\varphi : \mathcal{F} \upharpoonright M \rightarrow \mathbb{N}$ be hereditarily non constant in M . Let also $L \in [M]^\infty$ and $\mathcal{F}_{\varphi, L}$ be as in Lemma 3.24 and $g : \mathcal{F}_{\varphi, L} \rightarrow \mathbb{N}$ be an arbitrary map. For every $s \in \mathcal{F} \upharpoonright L$ let t_s be the unique initial segment of s in $\mathcal{F}_{\varphi, L}$. Then there exists $N \in [L]^\infty$ such that for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright N$, we have that

$$\varphi(s_2) - \varphi(s_1) > g(t_{s_1})$$

PROOF. By Proposition 1.12 there exist $N \in [L]^\infty$ such that either for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright N$, $\varphi(s_2) - \varphi(s_1) > g(t_{s_1})$ or for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright N$, $\varphi(s_2) - \varphi(s_1) \leq g(t_{s_1})$. We will show that the second case is impossible. Indeed, assume on the contrary. Then applying once more Proposition 1.12, we obtain $N_1 \in [N]^\infty$ such that either for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright N_1$, $|t_{s_1}| \leq |t_{s_2}|$ (the case $|t_{s_1}| > |t_{s_2}|$ is easily excluded).

Let (s_1^0, s_2^0) be a plegma pair in $\mathcal{F} \upharpoonright N_1$. We set $u_0 = t_{s_2^0} \setminus \{t_{s_2^0}\}$. We choose for every $n \in \mathbb{N}$ a plegma pair (s_1^n, s_2^n) of plegma pairs in $\mathcal{F} \upharpoonright N_1$ satisfying the following

- (a) $t_{s_1^0} \subseteq s_1^n$, for all $n \in \mathbb{N}$.
- (b) $u_0 \subseteq s_2^n$, for all $n \in \mathbb{N}$.
- (c) $t_{s_2^n} \neq t_{s_2^m}$, for all $n \neq m$.

These plegma pairs can be easily obtained as follows. Let $\tilde{s}_1 = s_1 \cap \{1, \dots, t_{s_2^0}\}$. We consider the following two cases.

Case 1: $|\tilde{s}_1| \leq |u_0|$, which yields that $s_1^0 = \tilde{s}_1$. In this case we set $s_1^n = s_1^0$ and $\tilde{s}_2^n = u_0 \cup \{m_n\}$ where $(m_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence in N_1 with $m_1 > \max(\tilde{s}_1 \cup u_0)$. Then we set s_2^n be any extension of \tilde{s}_2^n in $\mathcal{F} \upharpoonright N_1$.

Case 2: $|\tilde{s}_1| > |u_0|$, which yields that $|\tilde{s}_1| = |u_0| + 1$. We choose $(m_n)_{n \in \mathbb{N}}$ as in the previous case and we set $\tilde{s}_1^n = \tilde{s}_1$ and $\tilde{s}_2^n = u_0 \cup \{m_n\}$. We observe that $(\tilde{s}_1^n, \tilde{s}_2^n)$ is plegma with $|\tilde{s}_1^n| = |\tilde{s}_2^n|$ and we consider s_1^n, s_2^n extensions of $\tilde{s}_1^n, \tilde{s}_2^n$ respectively in $\mathcal{F} \upharpoonright N_1$ with (s_1^n, s_2^n) being a plegma pair.

Notice that $t_{s_1^n} \subseteq \tilde{s}_1 \subseteq s_1^n$ and $\tilde{s}_2^n = t_{s_2^n} \subseteq s_2^n$, for all $n \in \mathbb{N}$. Hence $\{\varphi(s_1^n) : n \in \mathbb{N}\}$ is a singleton and $\{\varphi(s_2^n) : n \in \mathbb{N}\}$ is infinite. Therefore there exists $n_0 \in \mathbb{N}$ such that $\varphi(s_2^n) > \varphi(s_1^1) + g(t_{s_1^1}) = \varphi(s_1^n) + g(t_{s_1^n})$, which is a contradiction and the proof is completed. \square

Corollary 3.26. Let \mathcal{F} be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $\varphi : \mathcal{F} \rightarrow \mathbb{N}$ be hereditarily non constant in M . Then there exists $N \in [M]^\infty$ such that for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright N$, $\varphi(s_2) - \varphi(s_1) > 1$.

3.3. Skipped Schauder Decompositions.

Definition 3.27. Let A be a countable seminormalized subset of a Banach space X . We say that A admits a *Skipped Schauder Decomposition* (SSD) if there exist $K > 0$ and a sequence $(F_n)_{n \in \mathbb{N}}$ of finite subsets of A such that

- (i) $\cup_{n \in \mathbb{N}} F_n = A$
- (ii) For every $L \in [\mathbb{N}]^\infty$ not containing two successive integers and every choice $(x_l)_{l \in L}$ such that $x_l \in F_l$, the sequence $(x_l)_{l \in L}$ is Schauder basic with basic constant K .

The following proposition is known and for the sake of completeness we outline its proof.

Proposition 3.28. Let $(x_n)_{n \in \mathbb{N}}$ be a seminormalized weakly null sequence in a Banach space X . Then for every $\varepsilon > 0$ the sequence $(x_n)_{n \in \mathbb{N}}$ admits a SSD with constant $1 + \varepsilon$.

PROOF. We assume that X has a Schauder basis $(e_n)_{n \in \mathbb{N}}$ with basis constant 1 (for example we may assume that $X = C[0, 1]$). We recursively define a partition $(F_n)_{n \in \mathbb{N}}$ of \mathbb{N} into finite pairwise disjoint sets and for every $n \in \mathbb{N}$ a finite block vector y_n such that the following are fulfilled:

- (i) If $n \in F_k$, $\|x_n - y_n\| < \frac{\varepsilon}{2^n}$.
- (ii) If $k + 1 < l$, $n \in F_k$ and $m \in F_l$ then $\max \text{supp}(y_n) < \min \text{supp}(y_m)$.

The induction uses the standard sliding hump argument. It is easy to check that $(F_k)_{k \in \mathbb{N}}$ is the desired SSD. \square

The following is a consequence of Definition 3.27 and Corollary 3.26.

Lemma 3.29. Let A be a subset of a Banach space X admitting a SSD with constant K . Let $M \in [\mathbb{N}]^\infty$, \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in A . Suppose that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is hereditarily nonconstant and $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model. Then $(e_n)_{n \in \mathbb{N}}$ is Schauder basic with constant K .

PROOF. Let $(F_k)_{k \in \mathbb{N}}$ be the partition of A witnessing its SSD property and $\varphi : \mathcal{F} \upharpoonright M \rightarrow \mathbb{N}$, defined by $\varphi(s) = k$ if $x_s \in F_k$. Observe that φ is hereditarily nonconstant in M . Indeed, assume on the contrary. Then there exists $N_1 \in [M]^\infty$ and $k_0 \in \mathbb{N}$ such that for every $s \in \mathcal{F} \upharpoonright N_1$, $x_s \in F_{k_0}$ for all $s \in \mathcal{F} \upharpoonright N_1$. Since F_{k_0} is finite, by Theorem 1.5 we get $N_2 \in [N_1]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright N_2}$ is constant. Therefore by Theorem 3.16, we get that $(e_n)_{n \in \mathbb{N}}$ is trivial which is a contradiction.

Since φ is hereditarily nonconstant in M , by Corollary 3.26 there exists $N \in [M]^\infty$ such that $\varphi(s_2) - \varphi(s_1) > 1$ for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright N$. Hence, by the SSD property of A we have that for every $l \in \mathbb{N}$ and for every plegma l -tuple $(s_j)_{j=1}^l$ in $\mathcal{F} \upharpoonright N$ the sequence $(x_{s_j})_{j=1}^l$ is Schauder basic with constant K . This easily yields that $(e_n)_{n \in \mathbb{N}}$ is Schauder basic with constant K . \square

Theorem 3.30. Let A be a subset of a Banach space X . If A admits a SSD with constant K , then every non trivial spreading model of any order in A is Schauder basic with constant K .

PROOF. Let \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in A . Let $M \in [\mathbb{N}]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates a non trivial \mathcal{F} -spreading model $(e_n)_{n \in \mathbb{N}}$. Then $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is hereditarily nonconstant. Indeed, otherwise there exists $L \in$

$[M]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is constant and therefore by Theorem 3.16 we get that $(e_n)_{n \in \mathbb{N}}$ is trivial. Hence the assumptions of Lemma 3.29 hold and the result follows. \square

4. Unconditional spreading models

In this section we provide a condition for a spreading model of any order to be unconditional. The key ingredient for establishing our results is the subordinated \mathcal{F} -sequences. The main result here is Theorem 3.32 which extends the well known theorem for classical spreading models generated by nontrivial weakly null sequences.

Lemma 3.31. Let X be a Banach space, $n \in \mathbb{N}$, $\mathcal{F}^1, \dots, \mathcal{F}^n$ be regular thin families, $L \in [\mathbb{N}]^\infty$ and $\varepsilon > 0$. For each $1 \leq i \leq n$, let $\widehat{\varphi}_i : \widehat{\mathcal{F}^i} \upharpoonright L \rightarrow (X, w)$ be a continuous map. Then there exist nonempty and finite $G^1 \subseteq \mathcal{F}^1 \upharpoonright L, \dots, G^n \subseteq \mathcal{F}^n \upharpoonright L$ and sequences $(\mu_t^1)_{t \in G^1}, \dots, (\mu_t^n)_{t \in G^n}$ in $[0, 1]$ such that

- (a) For all $1 \leq i \leq n$, $\sum_{t \in G^i} \mu_t^i = 1$ and $\|\widehat{\varphi}_i(\emptyset) - \sum_{t \in G^i} \mu_t^i \widehat{\varphi}_i(t)\| < \varepsilon$.
- (b) For every choice of $t_i \in G^i$, the n -tuple (t_1, \dots, t_n) is plegma.

PROOF. We use induction on $\xi_{\max} = \max\{o(\mathcal{F}^i) : 1 \leq i \leq n\}$. If $\xi_{\max} = 0$, i.e. $\mathcal{F}^i = \{\emptyset\}$ for all $1 \leq i \leq n$ the result follows trivially. (Let $G^i = \mathcal{F}^i$ and $\mu_\emptyset^i = 1$.) Let $1 \leq \xi \leq \omega_1$ and suppose that the conclusion of the proposition holds provided that $\xi_{\max} < \xi$. We will show that it also holds for $\xi_{\max} = \xi$. To this end let $n \in \mathbb{N}$, $L \in [\mathbb{N}]^\infty$ and $\mathcal{F}^1, \dots, \mathcal{F}^n$ be regular thin families such that $\xi = \max\{o(\mathcal{F}^i) : 1 \leq i \leq n\}$. Let also for each $1 \leq i \leq n$, $\widehat{\varphi}_i : \widehat{\mathcal{F}^i} \upharpoonright L \rightarrow (X, w)$ be a continuous map. We set $x_s^i = \widehat{\varphi}_i(s)$, for all $1 \leq i \leq n$ and $s \in \widehat{\mathcal{F}^i} \upharpoonright L$. We may suppose that \mathcal{F}^i is very large in L , therefore $\{l\} \in \widehat{\mathcal{F}^i}$ for all $l \in L$ and $1 \leq i \leq n$. Since for every $1 \leq i \leq n$ $w\text{-}\lim_{l \in L} x_{\{l\}}^i = x_\emptyset^i$, we can choose finite subsets $F^1 < \dots < F^n$ of L and sequences $(\lambda_l^1)_{l \in F^1}, \dots, (\lambda_l^n)_{l \in F^n}$ in $[0, 1]$ such that for every $i = 1, \dots, n$

- (i) $\sum_{l \in F^i} \lambda_l^i = 1$ and $\|x_\emptyset^i - \sum_{l \in F^i} \lambda_l^i x_{\{l\}}^i\| < \frac{\varepsilon}{2}$

Let $F = \cup_{i=1}^n F^i = \{l_1 < \dots < l_m\}$ and $L' = \{l \in L : l > l_m\}$. For every $1 \leq j \leq m$ let i_j be the unique $i \in \{1, \dots, n\}$ such that $l_j \in F^i$. For every $1 \leq j \leq n$ we define $\mathcal{G}^j = \mathcal{F}_{(l_j)}^{i_j}$ and $\widehat{\psi}_j : \widehat{\mathcal{G}^j} \upharpoonright L' \rightarrow (X, w)$, with $\widehat{\psi}_j(s) = \widehat{\varphi}_{i_j}(\{l_j\} \cup s)$. Since $o(\mathcal{G}^j) < o(\mathcal{F}^{i_j})$ we have that $\max\{o(\mathcal{G}^j) : 1 \leq j \leq m\} < \max\{o(\mathcal{F}^i) : 1 \leq i \leq n\}$ and therefore we may use the inductive assumption. Hence there exist nonempty finite subsets $\widehat{G}^1 \subseteq \widehat{\mathcal{G}^1} \upharpoonright L', \dots, \widehat{G}^m \subseteq \widehat{\mathcal{G}^m} \upharpoonright L'$ and sequences $(\widetilde{\mu}_s^1)_{s \in \widehat{G}^1}, \dots, (\widetilde{\mu}_s^m)_{s \in \widehat{G}^m}$ in $[0, 1]$ such that for every $1 \leq j \leq m$

- (ii) $\sum_{s \in \widehat{G}^j} \widetilde{\mu}_s^j = 1$ and $\|\widehat{\psi}_j(\emptyset) - \sum_{s \in \widehat{G}^j} \widetilde{\mu}_s^j \widehat{\psi}_j(s)\| < \frac{\varepsilon}{2}$.
- (iii) For every choice $t_j \in \widehat{G}^j$ the m -tuple (t_1, \dots, t_m) is plegma.

For every $1 \leq i \leq n$ we set $G^i = \{\{l_j\} \cup s : l_j \in F^i \text{ and } s \in \widehat{G}^j\}$ and for every $t \in G^i$ we set $\mu_t^i = \lambda_{l_j}^i \cdot \widetilde{\mu}_s^j$, where $t = \{l_j\} \cup s$. It is easy to see that

- (a) For all $1 \leq i \leq n$, $\sum_{t \in G^i} \mu_t^i = 1$.
- (b) For every choice of $t_i \in G^i$, the n -tuple (t_1, \dots, t_n) is plegma.

It remains to show that $\|\widehat{\varphi}_i(\emptyset) - \sum_{t \in G^i} \mu_t^i \widehat{\varphi}_i(t)\| < \varepsilon$, for all $1 \leq i \leq n$. Indeed

$$\begin{aligned} \|\widehat{\varphi}_i(\emptyset) - \sum_{t \in G^i} \mu_t^i \widehat{\varphi}_i(t)\| &\leq \|\widehat{\varphi}_i(\emptyset) - \sum_{\{j:i_j=i\}} \lambda_{l_j}^i \widehat{\varphi}_i(\{l_j\})\| \\ &\quad + \|\sum_{\{j:i_j=i\}} \lambda_{l_j}^i \widehat{\varphi}_i(\{l_j\}) - \sum_{t \in G^i} \mu_t^i \widehat{\varphi}_i(t)\| \end{aligned}$$

By (i) above we have that

$$\|\widehat{\varphi}_i(\emptyset) - \sum_{\{j:i_j=i\}} \lambda_{l_j}^i \widehat{\varphi}_i(\{l_j\})\| = \|x_\emptyset^i - \sum_{l \in F^i} \lambda_l^i x_{\{l\}}^i\| < \frac{\varepsilon}{2}$$

Moreover

$$\sum_{t \in G^i} \mu_t^i \widehat{\varphi}_i(t) = \sum_{\{j:i_j=i\}} \sum_{s \in \widetilde{G}^j} \lambda_{l_j}^i \widetilde{\mu}_s^j \widehat{\psi}_j(s)$$

Hence by (ii)

$$\|\sum_{\{j:i_j=i\}} \lambda_{l_j}^i \widehat{\varphi}_i(\{l_j\}) - \sum_{t \in G^i} \mu_t^i \widehat{\varphi}_i(t)\| \leq \sum_{\{j:i_j=i\}} \lambda_{l_j}^i \|\widehat{\psi}_j(\emptyset) - \sum_{s \in \widetilde{G}^j} \widetilde{\mu}_s^j \widehat{\psi}_j(s)\| < \frac{\varepsilon}{2}$$

And the proof is complete. \square

Theorem 3.32. Let \mathcal{F} be a regular thin family and $L \in [\mathbb{N}]^\infty$. Let $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ be an family of elements in a Banach space X generating a spreading model $(e_n)_{n \in \mathbb{N}}$. Suppose that $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is subordinated with respect to the weak topology on X and let $\widehat{\varphi} : \widehat{\mathcal{F}} \upharpoonright \widehat{L} \rightarrow (X, w)$ be the continuous map witnessing it. If $\widehat{\varphi}(\emptyset) = 0$, then either $(e_n)_{n \in \mathbb{N}}$ is trivial with $\|e_n\|_* = 0$ for all $n \in \mathbb{N}$ or the sequence $(e_n)_{n \in \mathbb{N}}$ is 1-unconditional (i.e. for every $n \in \mathbb{N}$, $F \subseteq \{1, \dots, n\}$ and $a_1, \dots, a_n \in \mathbb{R}$ we have that $\|\sum_{i \in F} a_i e_i\| \leq \|\sum_{i=1}^n a_i e_i\|$).

PROOF. Suppose that $(e_n)_{n \in \mathbb{N}}$ is trivial. Then by Theorem 3.16 there exists $L_1 \in [L]^\infty$ and $x_0 \in X$ such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L_1}$ converges to x_0 . By Proposition 3.10 we have that $x_0 = \widehat{\varphi}(\emptyset) = 0$. Since $(x_s)_{s \in \mathcal{F} \upharpoonright L_1}$ also generates $(e_n)_{n \in \mathbb{N}}$ we have that $\|e_1\|_* = 0$. The subsymmetry of $(e_n)_{n \in \mathbb{N}}$ yields that $\|e_n\|_* = 0$ for all $n \in \mathbb{N}$.

Assume that $(e_n)_{n \in \mathbb{N}}$ is trivial. In this case it is enough to show that for every $n \in \mathbb{N}$, $1 \leq p \leq n$ and $a_1, \dots, a_n \in [-1, 1]$ we have that

$$\left\| \sum_{\substack{i=1 \\ i \neq p}}^n a_i e_i \right\| \leq \left\| \sum_{i=1}^n a_i e_i \right\|$$

To this end it suffices to prove that for every $\varepsilon > 0$ we have that

$$\left\| \sum_{\substack{i=1 \\ i \neq p}}^n a_i e_i \right\| \leq \left\| \sum_{i=1}^n a_i e_i \right\| + \varepsilon$$

Indeed let $n \in \mathbb{N}$, $1 \leq p \leq n$, $a_1, \dots, a_n \in [-1, 1]$ and $\varepsilon > 0$. We easily pass to $N \in [L]^\infty$ such that for every plegma n -tuple $(s_i)_{i=1}^n$ in $\mathcal{F} \upharpoonright N$ the following hold

$$(5) \quad \left\| \sum_{i=1}^n a_i x_{s_i} \right\| - \left\| \sum_{i=1}^n a_i e_i \right\| \leq \frac{\varepsilon}{3} \quad \text{and} \quad \left\| \sum_{\substack{i=1 \\ i \neq p}}^n a_i x_{s_i} \right\| - \left\| \sum_{\substack{i=1 \\ i \neq p}}^n a_i e_i \right\| \leq \frac{\varepsilon}{3}$$

Let $\mathcal{F}^1 = \dots = \mathcal{F}^n = \mathcal{F}$ and $\widehat{\varphi}_1 = \dots = \widehat{\varphi}_n = \widehat{\varphi}|_{\mathcal{F} \upharpoonright N}$. By Lemma 3.31 there exist nonempty and finite $G^1, \dots, G^n \subseteq \mathcal{F} \upharpoonright N$ and sequences $(\mu_t^1)_{t \in G^1}, \dots, (\mu_t^n)_{t \in G^n}$ in $[0, 1]$ such that

$$(a) \text{ For } 1 \leq i \leq n, \sum_{t \in G^i} \mu_t^i = 1 \text{ and } \left\| \sum_{t \in G^i} \mu_t^i x_t \right\| = \|\widehat{\varphi}(\emptyset) - \sum_{t \in G^i} \mu_t^i \widehat{\varphi}(t)\| < \frac{\varepsilon}{3}.$$

(b) For every choice of $t_i \in G^i$, the n -tuple (t_1, \dots, t_n) is a plegma family.

We fix $s_i \in G^i$ for all $1 \leq i \leq n$ with $i \neq p$ and we set $G = G^p$. So we have that $\sum_{t \in G} \mu_t^p = 1$, $\left\| \sum_{t \in G} \mu_t^p x_t \right\| < \frac{\varepsilon}{3}$ and for every $t \in G$ the n -tuple $(s_1, \dots, s_{p-1}, t, s_{p+1}, \dots, s_n)$ is a plegma family. Therefore by (5) we have that

$$\begin{aligned} \left\| \sum_{\substack{i=1 \\ i \neq p}}^n a_i e_i \right\| &\leq \left\| \sum_{\substack{i=1 \\ i \neq p}}^n a_i x_{s_i} \right\| + \frac{\varepsilon}{3} \leq \left\| \sum_{\substack{i=1 \\ i \neq p}}^n a_i x_{s_i} + a_p \sum_{t \in G} \mu_t^p x_t \right\| + |a_p| \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &\leq \sum_{t \in G} \mu_t^p \left\| \sum_{\substack{i=1 \\ i \neq p}}^n a_i x_{s_i} + a_p x_t \right\| + \frac{2\varepsilon}{3} \leq \sum_{t \in G} \mu_t^p \left(\left\| \sum_{i=1}^n a_i e_i \right\| + \frac{\varepsilon}{3} \right) + \frac{2\varepsilon}{3} \\ &= \left\| \sum_{i=1}^n a_i e_i \right\| + \varepsilon \end{aligned}$$

and the proof is completed. \square

CHAPTER 4

Weakly relatively compact \mathcal{F} -sequences in Banach spaces

In the first section of this chapter we present some independent results concerning the behavior of sequences equivalent to the usual basis of ℓ^1 . All the other sections are devoted to the study of \mathcal{F} -sequences with weakly relatively compact range in Banach spaces with a Schauder basis. In this context, using the notion of subordinated \mathcal{F} -sequences, we show that every Schauder basic spreading model is an unconditional one. We also present the generic structural features of \mathcal{F} -sequences and finally we study the non Schauder basic spreading models.

1. ℓ^1 sequences

1.1. Splitting ℓ^1 sequences. In this subsection we study some stability properties of 1-subsymmetric sequences in seminormed spaces which are actually related to the non distortion of ℓ^1 (c.f. [17]). These results will be used in the next section.

Proposition 4.1. Let $(X, \|\cdot\|_\circ), (X_1, \|\cdot\|_*)$, $(X_2, \|\cdot\|_{**})$ be seminormed spaces having 1-subsymmetric Hamel bases $(e_n)_{n \in \mathbb{N}}, (e_n^1)_{n \in \mathbb{N}}$ and $(e_n^2)_{n \in \mathbb{N}}$ respectively. Suppose that $(e_n)_{n \in \mathbb{N}}$ admits lower ℓ^1 -estimate with constant $c > 0$ (i.e. for every $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$ we have that $c \sum_{i=1}^n |a_i| \leq \|\sum_{i=1}^n a_i e_i\|_\circ$). Assume also that

$$\left\| \sum_{i=1}^n a_i e_i \right\|_\circ \leq \left\| \sum_{i=1}^n a_i e_i^1 \right\|_* + \left\| \sum_{i=1}^n a_i e_i^2 \right\|_{**}$$

for all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$. If $(e_n^2)_{n \in \mathbb{N}}$ does not admit a lower ℓ^1 -estimate then $(e_n^1)_{n \in \mathbb{N}}$ admits a lower ℓ^1 -estimate with constant c .

PROOF. Suppose on the contrary that $(e_n^1)_{n \in \mathbb{N}}$ does not admit a lower ℓ^1 -estimate with constant c . Hence there exist $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$ with $\sum_{i=1}^n |a_i| = 1$ such that $\left\| \sum_{i=1}^n a_i e_i^1 \right\|_* < c - \varepsilon$, for some $\varepsilon > 0$. By the subsymmetry of $(e_n^1)_{n \in \mathbb{N}}$ we have that for every $k_1 < \dots < k_n$ in \mathbb{N}

$$\left\| \sum_{i=1}^n a_i e_{k_i}^1 \right\|_* < c - \varepsilon$$

Since $(e_n^2)_{n \in \mathbb{N}}$ does not admit a lower ℓ^1 -estimate, there exist $m \in \mathbb{N}$ and $b_1, \dots, b_m \in \mathbb{R}$ such that $\sum_{j=1}^m |b_j| = 1$ and $\left\| \sum_{j=1}^m b_j e_j^2 \right\|_{**} < \frac{\varepsilon}{2}$. Similarly by the subsymmetry of $(e_n^2)_{n \in \mathbb{N}}$ we have that for every $k_1 < \dots < k_m$ in \mathbb{N}

$$\left\| \sum_{j=1}^m b_j e_{k_j}^2 \right\|_{**} < \frac{\varepsilon}{2}$$

So we have the following two inequalities

$$\left\| \sum_{i=1}^n \sum_{j=1}^m a_i \cdot b_j e_{(i-1)m+j}^1 \right\|_* \leq \sum_{j=1}^m |b_j| \left\| \sum_{i=1}^n a_i e_{(i-1)m+j}^1 \right\|_* < \sum_{j=1}^m |b_j| \cdot (c - \varepsilon) = c - \varepsilon$$

and

$$\left\| \sum_{i=1}^n \sum_{j=1}^m a_i \cdot b_j e_{(i-1)m+j}^2 \right\|_{**} \leq \sum_{i=1}^n |a_i| \left\| \sum_{j=1}^m b_j e_{(i-1)m+j}^2 \right\|_{**} < \sum_{j=1}^m |b_j| \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$$

By the assumptions of the proposition we get that

$$\begin{aligned} \left\| \sum_{i=1}^n \sum_{j=1}^m a_i \cdot b_j e_{(i-1)m+j} \right\|_o &\leq \left\| \sum_{i=1}^n \sum_{j=1}^m a_i \cdot b_j e_{(i-1)m+j}^1 \right\|_* \\ &\quad + \left\| \sum_{i=1}^n \sum_{j=1}^m a_i \cdot b_j e_{(i-1)m+j}^2 \right\|_{**} < c - \frac{\varepsilon}{2} \end{aligned}$$

which since $\sum_{i=1}^n \sum_{j=1}^m |a_i| \cdot |b_j| = 1$, contradicts that $(e_n)_{n \in \mathbb{N}}$ admits lower a ℓ^1 -estimation with constant c . \square

Using the above proposition, it is easy to verify the following corollary.

Corollary 4.2. Let \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}, (x_s^1)_{s \in \mathcal{F}}, (x_s^2)_{s \in \mathcal{F}}$ be three \mathcal{F} -sequences in a Banach space X such that for all $s \in \mathcal{F}$, $x_s = x_s^1 + x_s^2$. Let $M \in [\mathbb{N}]^\infty$ and $(e_n)_{n \in \mathbb{N}}, (e_n^1)_{n \in \mathbb{N}}, (e_n^2)_{n \in \mathbb{N}}$ generated by $(x_s)_{s \in \mathcal{F} \upharpoonright M}, (x_s^1)_{s \in \mathcal{F} \upharpoonright M}$ and $(x_s^2)_{s \in \mathcal{F} \upharpoonright M}$ respectively as \mathcal{F} -spreading models. Suppose that $(e_n)_{n \in \mathbb{N}}$ admits a lower ℓ^1 -estimate with constant $c > 0$. If $(e_n^2)_{n \in \mathbb{N}}$ does not admit a lower ℓ^1 -estimate then $(e_n^1)_{n \in \mathbb{N}}$ admits a lower ℓ^1 -estimate with constant c .

1.2. Cesàro summability and ℓ^1 sequences. Recall that a sequence $(x_n)_{n \in \mathbb{N}}$ in a Banach space X is called *Cesàro summable to $x_0 \in X$* if

$$\frac{1}{n} \sum_{i=1}^n x_i \rightarrow x_0$$

The following result is well known but we include its proof for the sake of completeness.

Proposition 4.3. Let $(e_n)_{n \in \mathbb{N}}$ be an unconditional and subsymmetric sequence. Then either $(e_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^1 or $(e_n)_{n \in \mathbb{N}}$ is Cesàro summable to 0.

PROOF. Since $(e_n)_{n \in \mathbb{N}}$ is an unconditional and subsymmetric sequence, there exist $c_{\text{unc}}, c_{\text{sub}}, C_{\text{sub}} > 0$ such that for every $m \in \mathbb{N}$ and $a_1, \dots, a_m \in \mathbb{R}$,

$$c_{\text{unc}} \left\| \sum_{i=1}^m \varepsilon_i a_i e_i \right\| \leq \left\| \sum_{i=1}^m a_i e_i \right\|$$

for all $\varepsilon_1, \dots, \varepsilon_m \in \{-1, 1\}$ and

$$c_{\text{sub}} \left\| \sum_{i=1}^m a_i e_{k_i} \right\| \leq \left\| \sum_{i=1}^m a_i e_i \right\| \leq C_{\text{sub}} \left\| \sum_{i=1}^m a_i e_{k_i} \right\|$$

for all $k_1 < \dots < k_m$ in \mathbb{N} . Therefore, for $C = C_{\text{sub}} \|e_1\|$, we have that $\|e_n\| \leq C$, for all $n \in \mathbb{N}$.

Suppose that the sequence $(e_n)_{n \in \mathbb{N}}$ is not Cesàro summable to zero. We will show that $(e_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^1 . Since $(e_n)_{n \in \mathbb{N}}$ is not Cesàro summable to zero, there exist $\theta > 0$ and a strictly increasing sequence of natural numbers $(p_n)_{n \in \mathbb{N}}$ such that $\|\frac{1}{p_n} \sum_{i=1}^{p_n} e_i\| > \theta$, for all $i \in \{1, \dots, p_n\}$. Hence for every $n \in \mathbb{N}$ there exist x_n^* of norm 1 such that $x_n^*(\frac{1}{p_n} \sum_{i=1}^{p_n} e_i) > \theta$. For every $n \in \mathbb{N}$, we set $I_n = \{1, \dots, p_n\}$ and $A_n = \{i \in I_n : x_n^*(e_i) > \frac{\theta}{2}\}$. Hence for every $n \in \mathbb{N}$, we have that

$$\begin{aligned} \theta &< x_n^*\left(\frac{1}{p_n} \sum_{i \in I_n} e_i\right) = \frac{1}{p_n} x_n^*\left(\sum_{i \in A_n} e_i\right) + \frac{1}{p_n} x_n^*\left(\sum_{i \in I_n \setminus A_n} e_i\right) \\ &\leq \frac{1}{p_n} |A_n| C + \frac{\theta}{2} \end{aligned}$$

Hence $|A_n| \geq \frac{\theta}{2C} p_n \rightarrow \infty$. Let $m \in \mathbb{N}$ and $a_1, \dots, a_m \in \mathbb{R}$. It is immediate that

$$\left\| \sum_{i=1}^m a_i e_i \right\| \leq C \sum_{i=1}^m |a_i|$$

Since $|A_n| \rightarrow \infty$, there exists $n_0 \in \mathbb{N}$ such that $|A_{n_0}| \geq m$. Therefore we have that

$$\begin{aligned} \left\| \sum_{i=1}^m a_i e_i \right\| &\geq c_{\text{unc}} \left\| \sum_{i=1}^m |a_i| e_i \right\| \geq c_{\text{unc}} \cdot c_{\text{sub}} \left\| \sum_{i=1}^m |a_i| e_{A_{n_0}(i)} \right\| \\ &\geq c_{\text{unc}} \cdot c_{\text{sub}} \cdot x_n^* \left(\sum_{i=1}^m |a_i| e_{A_{n_0}(i)} \right) \geq c_{\text{unc}} \cdot c_{\text{sub}} \frac{\theta}{2} \sum_{i=1}^m |a_i| \end{aligned}$$

□

2. Spreading models generated by weakly relatively compact \mathcal{F} -sequences

Definition 4.4. Let X be a Banach space and $\xi < \omega_1$. By $\mathcal{SM}_\xi^{wrc}(X)$ we will denote the set of all 1-subsymmetric sequences $(e_n)_{n \in \mathbb{N}}$ such that there exists a weakly relatively compact subset A of X such that A admits $(e_n)_{n \in \mathbb{N}}$ as a ξ -order spreading model. We also set

$$\mathcal{SM}^{wrc}(X) = \bigcup_{\xi < \omega_1} \mathcal{SM}_\xi^{wrc}(X)$$

Since the weak topology on every countable weakly relatively compact subset of a Banach space is metrizable, by Proposition 3.11 we have the following.

Proposition 4.5. Let X be a Banach space, \mathcal{F} a regular thin family and $(x_s)_{s \in \mathcal{F}}$ a weakly relatively compact \mathcal{F} -sequence in X . Then for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is subordinated.

We will also need the following lemma, which is similar to the above proposition.

Lemma 4.6. Let X be a Banach space, $\xi < \omega_1$ and $(e_n)_{n \in \mathbb{N}} \in \mathcal{SM}_\xi^{wrc}(X)$. Then for every regular thin family \mathcal{F} of order ξ there exist $M \in [\mathbb{N}]^\infty$ and an \mathcal{F} -sequence $(x_s)_{s \in \mathcal{F}}$ in X satisfying the following:

- (i) $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model and
- (ii) for every $L' \in [M]^\infty$ there exists $L \in [L']^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is subordinated with respect to the weak topology.

PROOF. Let X be a Banach space, $\xi < \omega_1$ and $(e_n)_{n \in \mathbb{N}} \in \mathcal{SM}_\xi^{wrc}(X)$. Also let \mathcal{F} be a regular thin family of order ξ . Since $(e_n)_{n \in \mathbb{N}} \in \mathcal{SM}_\xi^{wrc}(X)$ there exists a weakly relatively compact subset A of X such that A admits $(e_n)_{n \in \mathbb{N}}$ as a ξ -order spreading model. Hence there exists a regular thin family \mathcal{G} of order ξ , $M' \in [\mathbb{N}]^\infty$ and an \mathcal{G} -sequence $(x'_t)_{t \in \mathcal{G}}$ in A such that $(x'_t)_{t \in \mathcal{G} \upharpoonright M'}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{G} -spreading model. By Proposition 2.13 there exist $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model and $\{x_s : s \in \mathcal{F}\} \subseteq \{x'_t : t \in \mathcal{G}\} \subseteq A$. Since the set A is weakly relatively compact, we get that the weak closure of $(x_s)_{s \in \mathcal{F}}$ is compact metrizable. By Proposition 3.11 we obtain that for every $L' \in [M]^\infty$ there exists $L \in [L']^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is subordinated with respect to the weak topology. \square

Lemma 4.7. Let $M \in [\mathbb{N}]^\infty$, \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in a Banach space X . Suppose that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is subordinated and let $\widehat{\varphi} : \widehat{\mathcal{F}} \upharpoonright M \rightarrow (X, w)$ be the continuous function witnessing the subordinating property of $(x_s)_{s \in \mathcal{F} \upharpoonright M}$. Assume that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates the sequence $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model. If $\widehat{\varphi}(\emptyset) \neq 0$ then either $(e_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^1 , or $(e_n)_{n \in \mathbb{N}}$ is not Schauder basic.

PROOF. We set $x'_s = x_s - \widehat{\varphi}(\emptyset)$, for all $s \in \mathcal{F} \upharpoonright M$. It is easy to see that the \mathcal{F} -subsequence $(x'_s)_{s \in \mathcal{F} \upharpoonright M}$ is subordinated as the map $\widehat{\varphi}' : \widehat{\mathcal{F}} \upharpoonright M \rightarrow (X, w)$, defined by $\widehat{\varphi}'(t) = \widehat{\varphi}(t) - \widehat{\varphi}(\emptyset)$ for all $t \in \widehat{\mathcal{F}} \upharpoonright M$, is continuous. Let $L \in [M]^\infty$ such that $(x'_s)_{s \in \mathcal{F} \upharpoonright L}$ generates an \mathcal{F} -spreading model $(e'_n)_{n \in \mathbb{N}}$. By Theorem 3.32 we have that either $(e'_n)_{n \in \mathbb{N}}$ is trivial and $\|e'_n\|_* = 0$ for all $n \in \mathbb{N}$, or $(e'_n)_{n \in \mathbb{N}}$ is an unconditional basic sequence.

In the first case by Definition 2.3 we have that $(x'_s)_{s \in \mathcal{F} \upharpoonright L}$ is norm convergent to zero. Hence the subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is norm convergent to $\widehat{\varphi}(\emptyset)$. By Theorem 3.16 we have that $(e_n)_{n \in \mathbb{N}}$ is trivial and consequently not Schauder basic.

In the second case since $(e'_n)_{n \in \mathbb{N}}$ is in addition 1- subsymmetric, by Proposition 4.3 we have that either $(e'_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^1 or $(e'_n)_{n \in \mathbb{N}}$ is Cesàro summable to zero. If $(e'_n)_{n \in \mathbb{N}}$ is equivalent to the basis of ℓ^1 , then by Corollary 4.2 it follows that $(e_n)_{n \in \mathbb{N}}$ is also equivalent to the basis of ℓ^1 . Suppose that $(e'_n)_{n \in \mathbb{N}}$ is Cesàro summable to zero. We will show that in this case $(e_n)_{n \in \mathbb{N}}$ is not Schauder basic. First notice that for every $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$ we have that

$$(6) \quad \left| \sum_{j=1}^n a_j \right| \cdot \|\widehat{\varphi}(\emptyset)\| - \left\| \sum_{j=1}^n a_j e'_j \right\| \leq \left\| \sum_{j=1}^n a_j e_j \right\| \leq \left| \sum_{j=1}^n a_j \right| \cdot \|\widehat{\varphi}(\emptyset)\| + \left\| \sum_{j=1}^n a_j e'_j \right\|$$

For every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\|\frac{1}{n} \sum_{j=1}^n e'_j\| < \varepsilon$. Hence by (6) we have that

$$\|\widehat{\varphi}(\emptyset)\| - \varepsilon \leq \left\| \frac{1}{n} \sum_{j=1}^n e_j \right\|$$

and

$$\left\| \frac{1}{n} \sum_{j=1}^n e_j - \frac{1}{n} \sum_{j=n+1}^{2n} e_j \right\| \leq 2\varepsilon$$

Since $\|\widehat{\varphi}(\emptyset)\| > 0$ we get that $(e_n)_{n \in \mathbb{N}}$ is not Schauder basic. \square

Theorem 4.8. Let X be a Banach space. Every Schauder basic sequence in $\mathcal{SM}^{wrc}(X)$ is unconditional.

PROOF. Let $\xi < \omega_1$ and $(e_n)_{n \in \mathbb{N}} \in \mathcal{SM}_\xi^{wrc}(X)$. Also let \mathcal{F} be a regular thin family of order ξ . Let $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in X satisfying the conclusion of Lemma 4.6. Let $L \in [M]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is subordinated and let $\widehat{\varphi} : \widehat{\mathcal{F}} \upharpoonright L \rightarrow (A, w)$ be the continuous map witnessing this fact. We distinguish two cases. Either $\widehat{\varphi}(\emptyset) = 0$, or $\widehat{\varphi}(\emptyset) \neq 0$. In the first case, by Theorem 3.32, we have that $(e_n)_{n \in \mathbb{N}}$ is 1-unconditional and in the second case, by Lemma 4.7 we have that $(e_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^1 . Hence in both cases we conclude that $(e_n)_{n \in \mathbb{N}}$ is unconditional. \square

Corollary 4.9. Let X be a Banach space, \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ be a weakly relatively compact \mathcal{F} -sequence which admits $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model. Then one of the following holds:

- (i) The sequence $(e_n)_{n \in \mathbb{N}}$ is trivial.
- (ii) The sequence $(e_n)_{n \in \mathbb{N}}$ is not trivial and not a Schauder basic one. In this case there exist $L \in [\mathbb{N}]^\infty$ and $x_0 \in X$ such that the \mathcal{F} -subsequence $(x'_s)_{s \in \mathcal{F} \upharpoonright L}$, defined by $x'_s = x_s - x_0$ for all $s \in \mathcal{F} \upharpoonright L$, generates an unconditional and Cesàro summable to zero \mathcal{F} -spreading model.
- (iii) The sequence $(e_n)_{n \in \mathbb{N}}$ is Schauder basic. In this case $(e_n)_{n \in \mathbb{N}}$ is unconditional.

PROOF. It is straightforward that Theorem 4.8 yields (iii). It remains to show that if the sequence $(e_n)_{n \in \mathbb{N}}$ is not trivial and not Schauder basic then (ii) holds. Let $M \in [\mathbb{N}]^\infty$ such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$. By Proposition 4.5 there exists $L_1 \in [M]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright L_1}$ is subordinated. Let $\widehat{\varphi} : \widehat{\mathcal{F}} \upharpoonright L_1 \rightarrow \overline{\{x_s : s \in \mathcal{F} \upharpoonright L_1\}}^w$ be the continuous map witnessing this. Since $(e_n)_{n \in \mathbb{N}}$ is not Schauder basic, by Theorem 3.32, we have that $\widehat{\varphi}(\emptyset) \neq 0$. We set $x_0 = \widehat{\varphi}(\emptyset)$. Let, for all $s \in \mathcal{F} \upharpoonright L_1$, $x'_s = x_s - x_0$ and let $L \in [L_1]^\infty$ such that the \mathcal{F} -subsequence $(x'_s)_{s \in \mathcal{F} \upharpoonright L}$ generates an \mathcal{F} -spreading model $(e'_n)_{n \in \mathbb{N}}$. Then the \mathcal{F} -subsequence $(x'_s)_{s \in \mathcal{F} \upharpoonright L_1}$ satisfies the assumptions of the Theorem 3.32. So $(e'_n)_{n \in \mathbb{N}}$ is unconditional. Since $(e_n)_{n \in \mathbb{N}}$ is not Schauder basic, it is not equivalent to the usual basis of ℓ^1 . Corollary 4.2 yields that $(e'_n)_{n \in \mathbb{N}}$ shares the same property and hence it is Cesàro summable to zero (Prop. 4.3). \square

3. The generic form of the weakly relatively compact \mathcal{F} -sequences

Let X be a Banach space with a Schauder basis. It is well known that a spreading model generated by a weakly null sequence $(x_n)_{n \in \mathbb{N}}$ in X is also generated by a block sequence $(\tilde{x}_n)_{n \in \mathbb{N}}$ which sufficiently approximates $(x_n)_{n \in \mathbb{N}}$. In this section we will show that analogues of this fact appear in spreading models which are generated by weakly relatively compact \mathcal{F} -sequences in X . To state our results we need to introduce the notions of *generic decompositions* and *generic assignments* to \mathcal{F} -sequences. We start with the first one.

Definition 4.10. Let X a Banach space with a Schauder basis. Let \mathcal{F} be a regular thin family and $(\tilde{x}_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in X . Let $L \in [\mathbb{N}]^\infty$ and $(\tilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L}$ be a family of vectors in X . We will say that $(\tilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L}$ is a generic decomposition of $(\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L}$ (or $(\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L}$ admits $(\tilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L}$ as a generic decomposition) if the following are satisfied.

- (i) For every $s \in \mathcal{F} \upharpoonright L$,

$$\tilde{x}_s = \sum_{k=0}^{|s|} \tilde{y}_{s|k} = \tilde{y}_\emptyset + \sum_{k=1}^{|s|} \tilde{y}_{s|k}$$

where some $\tilde{y}_{s|k}$ may be equal to zero, in which case $\text{supp}(\tilde{y}_{s|k}) = \emptyset$.

- (ii) For every $s \in \mathcal{F} \upharpoonright L$ and $1 \leq k_1 < k_2 \leq |s|$, $\text{supp}(\tilde{y}_{s|k_1}) < \text{supp}(\tilde{y}_{s|k_2})$.
 (iii) For every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L$ we have that
 (a) For every $1 \leq k_1 \leq |s_1|$ and $1 \leq k_2 \leq |s_2|$ with $k_1 \leq k_2$, $\text{supp}(\tilde{y}_{s_1|k_1}) < \text{supp}(\tilde{y}_{s_2|k_2})$.
 (b) For every $1 \leq k_1 < k_2 \leq |s_1|$, $\text{supp}(\tilde{y}_{s_2|k_1}) < \text{supp}(\tilde{y}_{s_1|k_2})$.

If in addition $\tilde{y}_\emptyset = 0$, then we will say that $(\tilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L}$ is a disjointly generic decomposition of $(\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L}$.

Proposition 4.11. Let X be a Banach space with a Schauder basis, \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in X . Let $M \in [\mathbb{N}]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is subordinated with respect to the weak topology and let $\widehat{\varphi} : \widehat{\mathcal{F}} \upharpoonright M \rightarrow (X, w)$ be the continuous map witnessing this. For every $s \in \mathcal{F} \upharpoonright M$ and $1 \leq k \leq |s|$, let

$$y_{s|k} = \widehat{\varphi}(s|k) - \widehat{\varphi}(s|k-1)$$

Let $d \in \mathbb{N}$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ be a decreasing sequence of positive reals. Then there exist $L \in [M]^\infty$, an \mathcal{F} -subsequence $(\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L}$ in X and a family $(\tilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L}$ of vectors in X such that the following are satisfied

- (i) $\tilde{y}_\emptyset = \widehat{\varphi}(\emptyset)$ and $(\tilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L}$ is a generic decomposition of $(\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L}$.
 (ii) For every $s \in \mathcal{F} \upharpoonright L$ with $\min s = L(n)$, $\|\tilde{x}_s - x_s\| < \varepsilon_n$.
 (iii) For every $s \in \mathcal{F} \upharpoonright L$ and $1 \leq k \leq |s|$, the following hold:
 (a) $d < \text{supp}(\tilde{y}_{s|k})$,
 (b) $\text{supp}(\tilde{y}_{s|k}) \subseteq \text{supp}(y_{s|k})$ and
 (c) if $\min s = L(n)$ then $\|\tilde{y}_{s|k} - y_{s|k}\| < \varepsilon_n/2^k$.
 (iv) For every $t \in \widehat{\mathcal{F}} \upharpoonright L \setminus \mathcal{F} \upharpoonright L$ if the set $N_t = \{l \in L : \tilde{y}_{t \cup \{l\}} \neq 0\}$ is infinite then $\min(\text{supp}(\tilde{y}_{t \cup \{l\}})) \xrightarrow{l \in N_t} \infty$.

PROOF. We pass to an $L_0 \in [M]^\infty$ such that \mathcal{F} is very large in L_0 . First we set $\tilde{y}_\emptyset = \widehat{\varphi}(\emptyset)$. By the continuity of $\widehat{\varphi}$ it is easy to see that for every $t \in \widehat{\mathcal{F}} \upharpoonright L_0 \setminus \mathcal{F} \upharpoonright L_0$, the sequence $(y_{t \cup \{l\}})_{l \in L_0}$ is weakly null. Using the standard sliding hump argument we construct $L_1 \in [L_0]^\infty$ and a block sequence $(\tilde{y}_{\{l\}})_{l \in L_1}$ such that for every $l \in L$ we have that

- (a) $d < \min(\text{supp}(\tilde{y}_{\{l\}}))$
 (b) $\text{supp}(\tilde{y}_{\{l\}}) \subseteq \text{supp}(y_{\{l\}})$ and
 (c) $\|\tilde{y}_{\{l\}} - y_{\{l\}}\| < \frac{\varepsilon_n}{2}$, where $L_1(n) = l$.

For every $t \in \widehat{\mathcal{F}} \upharpoonright L_1$ with $|t| \geq 2$ we define $l_t, k_t \in \mathbb{N}$ as follows

$$l_t = \min \left\{ l \in \mathbb{N} : \left\| \sum_{j=l+1}^{\infty} e_j^*(y_t) e_j \right\| < \frac{\varepsilon}{2 \cdot 2^{|t|}} \right\}$$

$$k_t = \max \left\{ k \in \mathbb{N} : k \leq l_t \text{ and } \left\| \sum_{j=l}^k e_j^*(y_t) e_j \right\| < \frac{\varepsilon}{2 \cdot 2^{|t|}} \right\}$$

where $(e_j^*)_{j \in \mathbb{N}}$ are the biorthogonals of $(e_j)_{j \in \mathbb{N}}$ and $(e_j)_{j \in \mathbb{N}}$ is the Schauder basis of X . If $k_t = l_t$ then we set $\tilde{y}_t = 0$. If $k_t < l_t$ then we set $\tilde{y}_t = \sum_{j=k_t}^{l_t} e_j^*(y_t)e_j$. For every $s \in \mathcal{F} \upharpoonright L_1$ we set

$$\tilde{x}_s = \sum_{k=0}^{|s|} y_{s|k}$$

It is easy to check that properties (iii) and (iv) are satisfied (with L_1 in place of L). Notice that (ii) is straightforward by (iii). Let

$$\mathcal{G} = \{s \in \mathcal{F} \upharpoonright L_1 : \text{supp}(\tilde{y}_{s|k_1}) < \text{supp}(\tilde{y}_{s|k_2}), \text{ for all } 1 \leq k_1 < k_2 \leq |s|\}$$

By (iv) and the compactness of $\widehat{\mathcal{F}}$ it is easy to see that \mathcal{G} is large in L_1 . Hence by Theorem 1.5 there exists $L_2 \in [L_1]^\infty$ such that $\mathcal{F} \upharpoonright L_2 \subseteq \mathcal{G}$ that is condition (ii) of the Definition 4.10 is satisfied.

Let \mathcal{A} be the set all plegma pairs in $\mathcal{F} \upharpoonright L_2$ satisfying condition (iii) of the Definition 4.10. Again we see that \mathcal{A} is large in L_2 and therefore by Corollary 1.13 there exists $L \in [L_2]^\infty$ such that every plegma pair in $\mathcal{F} \upharpoonright L$ belongs to \mathcal{A} . Finally, condition (i) of the Definition 4.10 is trivially satisfied and therefore the family $(y_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L}$ is a generic decomposition of $(\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L}$. \square

Notation 4.12. In the sequel we will say that the triple $(\widehat{\varphi}, (\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L}, (\tilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L})$ is a generic assignment to $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ (with respect to $(\varepsilon_n)_{n \in \mathbb{N}}$ and $d \in \mathbb{N}$).

Remark 4.13. Let us note that if $(\widehat{\varphi}, (\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L}, (\tilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L})$ is a generic assignment to $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ with respect to $(\varepsilon_n)_{n \in \mathbb{N}}$ and $d \in \mathbb{N}$, then the following are satisfied:

- (i) For every $L' \in [L]^\infty$ we have that $(\widehat{\varphi}, (\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L'}, (\tilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L'})$ is a generic assignment to $(x_s)_{s \in \mathcal{F} \upharpoonright L'}$.
- (ii) For every $(\varepsilon'_n)_{n \in \mathbb{N}}$ and every $d' \in \mathbb{N}$ there exists $L' \in [L]^\infty$ such that $(\widehat{\varphi}, (\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L'}, (\tilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L'})$ is a generic assignment to $(x_s)_{s \in \mathcal{F} \upharpoonright L'}$ with respect to $(\varepsilon'_n)_{n \in \mathbb{N}}$ and $d' \in \mathbb{N}$.
- (iii) $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model iff $(\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model.

The following is an easy consequence of Proposition 3.11 and Proposition 4.11.

Proposition 4.14. Let X be a Banach space with a Schauder basis, \mathcal{F} a regular thin family and $(x_s)_{s \in \mathcal{F}}$ a weakly relatively compact \mathcal{F} -sequence in X . Then for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ admits a generic assignment.

Theorem 4.15. Let X be a Banach space with a Schauder basis. Then for every $\xi < \omega_1$, every Schauder basic sequence $(e_n)_{n \in \mathbb{N}} \in \mathcal{SM}_\xi^{wrc}(X)$ and every regular thin family \mathcal{F} of order ξ , there exist an \mathcal{F} -sequence $(\tilde{x}_s)_{s \in \mathcal{F}}$ in X and $M \in [\mathbb{N}]^\infty$ such that $(\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright M}$ admits a disjointly generic decomposition and one of the following holds.

- (i) If $(e_n)_{n \in \mathbb{N}}$ is not equivalent to the usual basis of ℓ^1 , then $(\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model.
- (ii) If $(e_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^1 , then $(\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L}$ generates an \mathcal{F} -spreading model which is also equivalent to the usual basis of ℓ^1 .

PROOF. Let $\xi < \omega_1$, $(e_n)_{n \in \mathbb{N}}$ a Schauder basic sequence in $\mathcal{SM}_\xi^{wrc}(X)$ and regular thin family \mathcal{F} of order ξ . Then there exists a weakly relatively compact \mathcal{F} sequence $(x_s)_{s \in \mathcal{F}}$ and $L_0 \in [\mathbb{N}]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright L_0}$ generates $(e_n)_{n \in \mathbb{N}}$ as an

\mathcal{F} -spreading model. By Proposition 4.14 there exists $L_1 \in [L_0]^\infty$ such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L_1}$ admits a generic assignment $(\widehat{\varphi}, (\widetilde{x}_s)_{s \in \mathcal{F} \upharpoonright L_1}, (\widetilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L_1})$.

If $\widehat{\varphi}(\emptyset) = 0$ then (i) is immediate (see Remark 4.13). Suppose that $\widehat{\varphi}(\emptyset) \neq 0$. Since $(e_n)_{n \in \mathbb{N}}$ is Schauder basic, by Lemma 4.7 we have that $(e_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^1 . For every $s \in \mathcal{F} \upharpoonright L_1$ we set $x'_s = x_s - \widehat{\varphi}(\emptyset)$. Notice that the map $\widehat{\psi} : \widehat{\mathcal{F}} \upharpoonright L_1 \rightarrow (X, w)$ defined by $\widehat{\psi}(t) = \widehat{\varphi}(t) - \widehat{\varphi}(\emptyset)$ is continuous. Observe that $\widehat{\psi}(\emptyset) = 0$ and that $\widehat{\psi}(s) = x'_s$, for all $s \in \mathcal{F} \upharpoonright L_1$. By the continuity of $\widehat{\psi}$ we have that the \mathcal{F} -subsequence $(x'_s)_{s \in \mathcal{F} \upharpoonright L_1}$ is weakly relatively compact. Hence by Proposition 4.14 there exists $L_2 \in [L_1]^\infty$ such that the \mathcal{F} -subsequence $(x'_s)_{s \in \mathcal{F} \upharpoonright L_2}$ admits a generic assignment $(\widehat{\psi}, (\widetilde{x}'_s)_{s \in \mathcal{F} \upharpoonright L_2}, (\widetilde{y}'_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L_2})$. Let $L_3 \in [L_2]^\infty$ such that $(x'_s)_{s \in \mathcal{F} \upharpoonright L_3}$ generates $(e'_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model. By Corollary 4.2 $(e'_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^1 . Hence by Remark 4.13 we get (ii). \square

4. Singular spreading models

Let X be a Banach space with a Schauder basis of constant $C > 0$. Let \mathcal{F} be regular thin, $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in X satisfying the following:

- (i) The \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates an \mathcal{F} -spreading model $(e_n)_{n \in \mathbb{N}}$.
- (ii) The \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is subordinated and let $\widehat{\varphi} : \widehat{\mathcal{F}} \upharpoonright M \rightarrow (X, w)$ be the continuous map witnessing this.
- (iii) The \mathcal{F} -subsequence $(x'_s)_{s \in \mathcal{F} \upharpoonright M}$, defined by $x'_s = x_s - \widehat{\varphi}(\emptyset)$ for all $s \in \mathcal{F} \upharpoonright M$, generates an \mathcal{F} -spreading model $(e'_n)_{n \in \mathbb{N}}$.

The aim of this section is to study the relation between $(e_n)_{n \in \mathbb{N}}$ and $(e'_n)_{n \in \mathbb{N}}$. We start with the following.

Proposition 4.16. Let $\|\cdot\|_*$ (resp. $\|\cdot\|'_*$) be the seminorm defined on $(e_n)_{n \in \mathbb{N}}$ (resp. $(e'_n)_{n \in \mathbb{N}}$). The following are equivalent:

- (i) The sequence $(e_n)_{n \in \mathbb{N}}$ is trivial.
- (ii) $\|z\|'_* = 0$ for all $z \in \langle (e'_n)_{n \in \mathbb{N}} \rangle$.
- (iii) The sequence $(e'_n)_{n \in \mathbb{N}}$ is trivial.

PROOF. (i) \Rightarrow (ii): Suppose that $(e_n)_{n \in \mathbb{N}}$ is trivial. By Theorem 3.16 there exists $x_0 \in X$ and $L \in [M]^\infty$ such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ norm converges to x_0 . Thus $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ weakly converges to x_0 and therefore by Proposition 3.10 we have that $x_0 = \widehat{\varphi}(\emptyset)$. Hence the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ norm converges to $\widehat{\varphi}(\emptyset)$. This easily yields that $(x'_s)_{s \in \mathcal{F} \upharpoonright L}$ norm converges to 0_X , which implies that $\|z\|'_* = 0$ for all $z \in \langle (e'_n)_{n \in \mathbb{N}} \rangle$.

(ii) \Rightarrow (iii): It is obvious.

(iii) \Rightarrow (i): Assume that $\|z\|'_* = 0$, for all $z \in \langle (e'_n)_{n \in \mathbb{N}} \rangle$. Then the sequence $(e'_n)_{n \in \mathbb{N}}$ is trivial and by Theorem 3.16 there exists $L \in [M]^\infty$ such that the \mathcal{F} -subsequence $(x'_s)_{s \in \mathcal{F} \upharpoonright L}$ is norm Cauchy. This easily yields that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is also norm Cauchy and again by Theorem 3.16 we have that $(e_n)_{n \in \mathbb{N}}$ is trivial. \square

For the rest of this section we assume that the sequence $(e_n)_{n \in \mathbb{N}}$ is nontrivial. Therefore by the above proposition we have that the sequence $(e'_n)_{n \in \mathbb{N}}$ is also nontrivial. Let

$$E = \overline{\langle (e_n)_{n \in \mathbb{N}} \rangle}^{\|\cdot\|_*} \quad \text{and} \quad E' = \overline{\langle (e'_n)_{n \in \mathbb{N}} \rangle}^{\|\cdot\|'_*}$$

If $\widehat{\varphi}(\emptyset) = 0$ then $(e_n)_{n \in \mathbb{N}} = (e'_n)_{n \in \mathbb{N}}$ and $\|\cdot\|_* = \|\cdot\|'_*$.

Lemma 4.17. Suppose that $\widehat{\varphi}(\emptyset) \neq 0$. Then E is isomorphic to a subspace of $Z = (\mathbb{R} \oplus E')_0$.

PROOF. Let $T : (e_n)_{n \in \mathbb{N}} \rightarrow Z$ defined by

$$T\left(\sum_{j=1}^n a_j e_j\right) = \left(\sum_{j=1}^n a_j, \sum_{j=1}^n a_j e'_j\right)$$

for all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$. We will show that for all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$

$$K_1 \left\| T\left(\sum_{j=1}^n a_j e_j\right) \right\|_Z \leq \left\| \sum_{j=1}^n a_j e_j \right\|_* \leq K_2 \left\| T\left(\sum_{j=1}^n a_j e_j\right) \right\|_Z$$

where $K_1 = \frac{\min\{1, \|\widehat{\varphi}(\emptyset)\|\}}{2C}$ and $K_2 = 2 \max\{1, \|\widehat{\varphi}(\emptyset)\|\}$.

By Proposition 4.11 there exists $L \in [M]^\infty$ such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ admits a generic assignment $(\widehat{\varphi}, (\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L}, (\tilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L})$ with respect to a decreasing null sequence of reals $(\varepsilon_k)_{k \in \mathbb{N}}$. Let $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$. We can easily choose a sequence $(s_k)_{k \in \mathbb{N}}$ in $\mathcal{F} \upharpoonright L$ such that for every $k \in \mathbb{N}$ the n -tuple $(s_{k+j})_{j=1}^n$ is plegma. For every $k \in \mathbb{N}$, let $l_k = \min(\text{supp}(\tilde{x}_{s_k} - \widehat{\varphi}(\emptyset)))$. Notice that $l_k \nearrow \infty$ and $P_{\{1, \dots, l_k-1\}}(\tilde{x}_{s_k} - \widehat{\varphi}(\emptyset)) = 0$. Hence

$$\begin{aligned} \left\| \sum_{j=1}^n a_j e_j \right\|_* &= \lim_{k \rightarrow \infty} \left\| \sum_{j=1}^n a_j x_{s_{k+j}} \right\| \leq \left\| \sum_{j=1}^n a_j \widehat{\varphi}(\emptyset) \right\| + \lim_{k \rightarrow \infty} \left\| \sum_{j=1}^n a_j x'_{s_{k+j}} \right\| \\ &= \left| \sum_{j=1}^n a_j \right| \cdot \|\widehat{\varphi}(\emptyset)\| + \left\| \sum_{j=1}^n a_j e'_j \right\|'_* \leq 2 \max\{1, \|\widehat{\varphi}(\emptyset)\|\} \left\| T\left(\sum_{j=1}^n a_j e_j\right) \right\|_Z \end{aligned}$$

On the other hand for every $k \in \mathbb{N}$ we have that

$$\begin{aligned} \left\| \sum_{j=1}^n a_j x_{s_{k+j}} \right\| &\geq \left\| \sum_{j=1}^n a_j \tilde{x}_{s_{k+j}} \right\| - \varepsilon_k \left| \sum_{j=1}^n a_j \right| \\ &\geq \frac{1}{C} \|P_{\{1, \dots, l_k-1\}}(\widehat{\varphi}(\emptyset))\| \cdot \left| \sum_{j=1}^n a_j \right| - \varepsilon_k \left| \sum_{j=1}^n a_j \right| \end{aligned}$$

Hence

$$\left\| \sum_{j=1}^n a_j e_j \right\|_* \geq \frac{1}{C} \|\widehat{\varphi}(\emptyset)\| \cdot \left| \sum_{j=1}^n a_j \right| \geq \frac{1}{2C} \min\{1, \|\widehat{\varphi}(\emptyset)\|\} \cdot \left| \sum_{j=1}^n a_j \right|$$

For every $k \in \mathbb{N}$ we also have that

$$\begin{aligned} \left\| \sum_{j=1}^n a_j x_{s_{k+j}} \right\| &\geq \frac{1}{2C} \left\| P_{\{l_k, \dots\}} \left(\sum_{j=1}^n a_j x_{s_{k+j}} \right) \right\| \\ &\geq \frac{1}{2C} \left\| P_{\{l_k, \dots\}} \left(\sum_{j=1}^n a_j x'_{s_{k+j}} \right) \right\| - \frac{1}{2C} \left| \sum_{j=1}^n a_j \right| \cdot \|P_{\{l_k, \dots\}}(\widehat{\varphi}(\emptyset))\| \end{aligned}$$

and

$$\begin{aligned}
\left\| P_{\{l_k, \dots\}} \left(\sum_{j=1}^n a_j x'_{s_{k+j}} \right) \right\| &\geq \left\| \sum_{j=1}^n a_j x'_{s_{k+j}} \right\| - \left\| P_{\{1, \dots, l_k-1\}} \left(\sum_{j=1}^n a_j x'_{s_{k+j}} \right) \right\| \\
&\geq \left\| \sum_{j=1}^n a_j x'_{s_{k+j}} \right\| - 2C \left| \sum_{j=1}^n a_j \right| \varepsilon_k - \left\| P_{\{1, \dots, l_k-1\}} \left(\sum_{j=1}^n a_j (\tilde{x}'_{s_{k+j}} - \widehat{\varphi}(\emptyset)) \right) \right\| \\
&= \left\| \sum_{j=1}^n a_j x'_{s_{k+j}} \right\| - 2C \left| \sum_{j=1}^n a_j \right| \varepsilon_k
\end{aligned}$$

Hence

$$\left\| \sum_{j=1}^n a_j x_{s_{k+j}} \right\| \geq \frac{1}{2C} \left\| \sum_{j=1}^n a_j x'_{s_{k+j}} \right\| - \left| \sum_{j=1}^n a_j \right| \varepsilon_k - \frac{1}{2C} \left| \sum_{j=1}^n a_j \right| \cdot \|P_{\{l_k, \dots\}}(\widehat{\varphi}(\emptyset))\|$$

Therefore

$$\left\| \sum_{j=1}^n a_j e_j \right\|_* \geq \frac{1}{2C} \left\| \sum_{j=1}^n a_j e'_j \right\|'_* \geq \frac{1}{2C} \min\{1, \|\widehat{\varphi}(\emptyset)\|\} \left\| \sum_{j=1}^n a_j e'_j \right\|'_*$$

Summarizing the above we get

$$\frac{\min\{1, \|\widehat{\varphi}(\emptyset)\|\}}{2C} \left\| T \left(\sum_{j=1}^n a_j e_j \right) \right\|_Z \leq \left\| \sum_{j=1}^n a_j e_j \right\|_*$$

□

The next proposition is the main result of this section.

Proposition 4.18. Every Schauder basic sequence in E has a block subsequence which is equivalent to a block subsequence of $(e'_n)_{n \in \mathbb{N}}$.

PROOF. If $\widehat{\varphi}(\emptyset) = 0$ then $(e_n)_{n \in \mathbb{N}} \equiv (e'_n)_{n \in \mathbb{N}}$ and the result follows easily by the sliding hump argument.

Suppose that $\widehat{\varphi}(\emptyset) \neq 0$. By the proof of Lemma 4.17 we have that the map $T : E \rightarrow Z = (\mathbb{R} \oplus E')_0$ with $T(\sum_{j=1}^n a_j e_j) = (\sum_{j=1}^n a_j, \sum_{j=1}^n a_j e'_j)$ is an isomorphic embedding of E into Z .

Let $(v_n)_{n \in \mathbb{N}}$ be a basic sequence in E with basis constant c . Let $(\delta_n)_{n \in \mathbb{N}}$ be a sequence of positive reals such that $\sum_{j=1}^\infty \delta_j < \frac{1}{2c}$. Let $(\tilde{v}_n)_{n \in \mathbb{N}}$ be a sequence in $\langle (e_n)_{n \in \mathbb{N}} \rangle$ such that $\|v_n - \tilde{v}_n\| < \delta_n$ for all $n \in \mathbb{N}$. Then the sequences $(v_n)_{n \in \mathbb{N}}$ and $(\tilde{v}_n)_{n \in \mathbb{N}}$ are equivalent.

Let $(a_n, w_n) = T(\tilde{v}_n)$ for all $n \in \mathbb{N}$. Since T is an isomorphic embedding, the sequences $((a_n, w_n))_{n \in \mathbb{N}}$ and $(\tilde{v}_n)_{n \in \mathbb{N}}$ are equivalent Schauder basic sequences. Let $c' > 0$ be the basis constant of $((a_n, w_n))_{n \in \mathbb{N}}$ and $(\delta'_n)_{n \in \mathbb{N}}$ a sequence of positive reals such that $\sum_{n=1}^\infty \delta'_n < \frac{1}{2c'}$. Using the standard sliding hump argument we may choose sequences $(k_n)_{n \in \mathbb{N}}$ and $(l_n)_{n \in \mathbb{N}}$ of natural numbers and a sequence $(F_n)_{n \in \mathbb{N}}$ of finite subsets of \mathbb{N} such that for every $n \in \mathbb{N}$ the following are satisfied

- (i) $k_n < l_n < k_{n+1}$,
- (ii) $\max F_n < \min F_{n+1}$,
- (iii) $|a_{l_n} - a_{k_n}| < \delta'_n$ and
- (iv) $\|P_{F_n^c}(w_{l_n} - w_{k_n})\|'_* < \delta'_n$.

For every $n \in \mathbb{N}$ we set $z_n = P_{F_n}(w_{l_n} - w_{k_n})$. It is obvious that the sequences $(z_n)_{n \in \mathbb{N}}$ and $((0, z_n))_{n \in \mathbb{N}}$ are equivalent. For every $n \in \mathbb{N}$ we have that

$$\begin{aligned} \|(0, z_n) - ((a_{l_n}, w_{l_n}) - (a_{k_n}, w_{k_n}))\|_Z &= \|(0, z_n) - (a_{l_n} - a_{k_n}, w_{l_n} - w_{k_n})\|_Z \\ &= \max\{|a_{l_n} - a_{k_n}|, \|z_n - (w_{l_n} - w_{k_n})\|'_*\} \\ &= \max\{|a_{l_n} - a_{k_n}|, \|P_{F_n^c}(w_{l_n} - w_{k_n})\|'_*\} < \delta'_n \end{aligned}$$

Thus the sequences $((0, z_n))_{n \in \mathbb{N}}$ and $(T(\tilde{v}_{l_n} - \tilde{v}_{k_n}))_{n \in \mathbb{N}} = ((a_{l_n}, w_{l_n}) - (a_{k_n}, w_{k_n}))_{n \in \mathbb{N}}$ are equivalent. This implies that $(z_n)_{n \in \mathbb{N}}$ and $(v_{l_n} - v_{k_n})_{n \in \mathbb{N}}$ are also equivalent. \square

CHAPTER 5

Composition of the spreading models

In this chapter we present some composition properties for spreading models. Among others we show that the class of higher order spreading models introduced in [24] are also spreading models of the same order in our context. Moreover, we present several related results for ℓ^p and c_0 spreading models.

1. The composition property

Definition 5.1. Let X be a Banach space with a Schauder basis. Let \mathcal{F} be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in X consisted by finite supported vectors. We will say that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is plegma disjointly supported (resp. plegma block) if for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright M$ we have that $\text{supp}(x_{s_1}) \cap \text{supp}(x_{s_2}) = \emptyset$ (resp. $\text{supp}(x_{s_1}) < \text{supp}(x_{s_2})$).

Definition 5.2. Let X be a Banach space with a Schauder basis, \mathcal{F} a regular thin family and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in X consisted by finite supported vectors. Let $M \in [\mathbb{N}]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates an \mathcal{F} -spreading model $(e_n)_{n \in \mathbb{N}}$. We will say that $(e_n)_{n \in \mathbb{N}}$ is plegma disjointly (resp. plegma block) generated by $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ if $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is plegma disjointly supported (resp. plegma block).

Definition 5.3. Let \mathcal{F}, \mathcal{G} be families of finite subsets of \mathbb{N} . We define the ordered direct sum of \mathcal{F} and \mathcal{G} to be the family

$$\mathcal{G} \oplus \mathcal{F} = \left\{ s \cup t : s \in \mathcal{G}, t \in \mathcal{F} \text{ and } s < t \right\}$$

Remark 5.4. It is easy to see that if \mathcal{F} and \mathcal{G} are regular thin families, then $\mathcal{G} \oplus \mathcal{F}$ is also a regular thin family and $o(\mathcal{G} \oplus \mathcal{F}) = o(\mathcal{F}) + o(\mathcal{G})$. In particular $o([\mathbb{N}]^k \oplus \mathcal{F}) = o(\mathcal{F}) + k$, for every $k \in \mathbb{N}$.

Theorem 5.5. Let X be a Banach space and $(e_n)_{n \in \mathbb{N}}$ be a Schauder basic sequence in $\mathcal{SM}_\xi(X)$, for some $\xi < \omega_1$. Let $E = \langle (e_n)_{n \in \mathbb{N}} \rangle$ and for some $k \in \mathbb{N}$, let $(\bar{e}_n)_{n \in \mathbb{N}} \in \mathcal{SM}_k(E)$ be a plegma block generated spreading model. Then

$$(\bar{e}_n)_{n \in \mathbb{N}} \in \mathcal{SM}_{\xi+k}(X)$$

PROOF. Let \mathcal{F} be a regular thin family and $o(\mathcal{F}) = \xi$, $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in X and $M \in [\mathbb{N}]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$. We may also assume that \mathcal{F} is very large in M . Let $(y_t)_{t \in [\mathbb{N}]^k}$ in $\langle (e_n)_{n \in \mathbb{N}} \rangle$ which admits $(\bar{e}_n)_{n \in \mathbb{N}}$ as a plegma block generated $[\mathbb{N}]^k$ -spreading model. By Remark 2.5 we may suppose that whole $(y_t)_{t \in [\mathbb{N}]^k}$ plegma block generates $(\bar{e}_n)_{n \in \mathbb{N}}$ as an $[\mathbb{N}]^k$ -spreading model. For every $t \in [\mathbb{N}]^k$ we set $F_t = \text{supp}(y_t)$ with respect to $(e_n)_{n \in \mathbb{N}}$ and $l_t = |F_t|$. Then for every $t \in [\mathbb{N}]^k$ the vector y_t is of the form

$$y_t = \sum_{j=1}^{l_t} a_{F_t(j)}^t e_{F_t(j)}$$

We set $\mathcal{G} = [\mathbb{N}]^k \oplus \mathcal{F}$ and for every $v \in \mathcal{G}$ we set t_v and s_v the unique elements in $[\mathbb{N}]^k$ and \mathcal{F} respectively such that $v = t_v \cup s_v$ and $t_v < s_v$. We split the proof into three steps.

Step 1: We set

$$\mathcal{G}^* = \left\{ v \in \mathcal{G} : \min s_v \geq M(l_{t_v}) \right\}$$

It is easy to verify that \mathcal{G}^* is large in M . Hence by Theorem 1.5 there exists $L_0 \in [M]^\infty$ such that $\mathcal{G} \upharpoonright L_0 \subseteq \mathcal{G}^*$. For every $v \in \mathcal{G} \upharpoonright L_0$ we define a finite sequence $(s_1^v, \dots, s_{l_{t_v}}^v)$ as follows. Let $s_v = \{M(q_1^v), \dots, M(q_{|s_v|}^v)\}$. Then for every $j = 1, \dots, l_{t_v}$ we set s_j^v to be the unique element of $\mathcal{F} \upharpoonright M$ such that

$$s_j^v \subseteq \left\{ M(q_i^v - l_{t_v} + j) : i = 1, \dots, |s_v| \right\}$$

We define the family $(z_v)_{v \in \mathcal{G} \upharpoonright L_0}$ by setting

$$z_v = \sum_{j=1}^{l_{t_v}} a_{F_{t_v}(j)}^{t_v} x_{s_j^v}$$

for every $v \in \mathcal{G} \upharpoonright L_0$.

Step 2: For every $n \in \mathbb{N}$ we set

$$\begin{aligned} \mathcal{A}_n = \left\{ (v_p)_{p=1}^n \in Plm_n(\mathcal{G} \upharpoonright L_0) : s_1^{v_1}(1) \geq M\left(\sum_{p=1}^n l_{v_p}\right) \right. \\ \left. \text{and } (s_j^{v_1})_{j=1}^{l_{t_{v_1}}} \cap \dots \cap (s_j^{v_n})_{j=1}^{l_{t_{v_n}}} \text{ is plegma} \right\} \end{aligned}$$

It is easily verified that \mathcal{A}_n is large in L_0 . Inductively using Proposition 1.12 we construct a decreasing sequence $(L_n)_{n \in \mathbb{N}}$ in $[L_0]^\infty$ such that $Plm_n(\mathcal{G} \upharpoonright L_n) \subseteq \mathcal{A}_n$ for all $n \in \mathbb{N}$. Let L be a diagonalization of $(L_n)_{n \in \mathbb{N}}$, that is $L(n) \in L_n$ for each n .

Step 3: In this step we will show that $(z_s)_{s \in \mathcal{G} \upharpoonright L}$ admits $(\bar{e}_n)_{n \in \mathbb{N}}$ as a $\xi + k$ order spreading model. Indeed, first notice that the family \mathcal{G} is of order $\xi + k$. Let $(\delta_n^1)_{n \in \mathbb{N}}$ be a null sequence of positive numbers such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$ with respect to $(\delta_n^1)_{n \in \mathbb{N}}$. Let also $(\delta_n^2)_{n \in \mathbb{N}}$ be a null sequence of positive numbers such that $(y_t)_{t \in [L]^k}$ generates $(\bar{e}_n)_{n \in \mathbb{N}}$ with respect to $(\delta_n^2)_{n \in \mathbb{N}}$. Let C be the basis constant of $(e_n)_{n \in \mathbb{N}}$, $K = \sup\{\|y_t\| : t \in [L]^k\}$ and set $\delta_n = 2CK\delta_n^1 + \delta_n^2$, $n \in \mathbb{N}$.

We will show that $(z_v)_{v \in \mathcal{G} \upharpoonright L}$ satisfies the conditions of Remark 2.6 concerning the sequence $(\bar{e}_n)_{n \in \mathbb{N}}$ with respect to $(\delta_n)_{n \in \mathbb{N}}$ and therefore there exists $L' \in [L]^\infty$ such that $(z_v)_{v \in \mathcal{G} \upharpoonright L'}$ generates $(\bar{e}_n)_{n \in \mathbb{N}}$ as a \mathcal{G} -spreading model. Let $l \in \mathbb{N}$, $(v_i)_{i=1}^l$ a plegma m -tuple in $\mathcal{G} \upharpoonright L$ with $v_1(1) \geq L(l)$ and $b_1, \dots, b_l \in [-1, 1]$. By Step 2, $(v_i)_{i=1}^l$ belongs to \mathcal{A}_l . Notice that

$$\begin{aligned} (7) \quad \left\| \sum_{i=1}^l b_i z_{v_i} \right\| - \left\| \sum_{i=1}^l b_i \bar{e}_i \right\| &\leq \left\| \sum_{i=1}^l b_i z_{v_i} \right\| - \left\| \sum_{i=1}^l b_i y_{t_{v_i}} \right\| \\ &\quad + \left\| \sum_{i=1}^l b_i y_{t_{v_i}} \right\| - \left\| \sum_{i=1}^l b_i \bar{e}_i \right\| \end{aligned}$$

It is straightforward that

$$(8) \quad \left\| \sum_{i=1}^l b_i y_{t_{v_i}} \right\| - \left\| \sum_{i=1}^l b_i \bar{e}_i \right\| < \delta_l^2$$

Since $(e_n)_{n \in \mathbb{N}}$ is Schauder basic with basis constant C and for every $t \in [L]^k$, $\|y_t\| \leq K$, we have that for every $t \in [L]^k$ and $1 \leq j \leq l_t$, $|a_{F_t(j)}^t| \leq 2CK$. Hence for every $1 \leq i \leq l$, $t \in [L]^k$ and $1 \leq j \leq l_t$, we have that $b_i a_{F_t(j)}^t \in [-2CK, 2CK]$. Since $(v_i)_{i=1}^l$ belongs to \mathcal{A}_l we have that $v_1(1) \geq M(\sum_{i=1}^l l_{v_i}) \geq L(\sum_{i=1}^l l_{v_i})$. So

$$(9) \quad \left\| \sum_{i=1}^l b_i z_{v_i} \right\| - \left\| \sum_{i=1}^l b_i y_{t_{v_i}} \right\| = \left\| \sum_{i=1}^l \sum_{j=1}^{l_{v_i}} b_i a_{F_{t_{v_i}}(j)}^{t_{v_i}} x_j^{s_{v_i}} \right\| - \left\| \sum_{i=1}^l \sum_{j=1}^{l_{v_i}} b_i a_{F_{t_{v_i}}(j)}^{t_{v_i}} e_{F_{t_{v_i}}(j)} \right\| < 2CK\delta_l^1$$

The inequalities (7), (8) and (9) yield that

$$\left\| \sum_{i=1}^l b_i z_{v_i} \right\| - \left\| \sum_{i=1}^l b_i \bar{e}_i \right\| < \delta_l^2 + 2CK\delta_l^1 = \delta_l$$

Hence by Remark 2.6, we get that for some $L' \in [L]^\infty$, $(z_v)_{v \in \mathcal{G} \upharpoonright L'}$ generates $(\bar{e}_n)_{n \in \mathbb{N}}$ as a \mathcal{G} -spreading model. \square

Remark 5.6. The above proof actually gives more information concerning the structure of the sequence $(z_v)_{v \in \mathcal{G} \upharpoonright L}$. First notice that by Step 2, we have that if we additionally assume that X has a Schauder basis then

- (i) If $(e_n)_{n \in \mathbb{N}}$ is plegma disjointly (resp. plegma block) generated by $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ then $(\bar{e}_n)_{n \in \mathbb{N}}$ is plegma disjointly (resp. plegma block) generated by $(z_v)_{v \in \mathcal{G} \upharpoonright L'}$.
- (ii) If $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ admits a disjointly generic decomposition then $(z_v)_{v \in \mathcal{G} \upharpoonright L}$ also does.
- (iii) For every $v \in \mathcal{G} \upharpoonright L$ there exist $m \in \mathbb{N}$ and $s_1, \dots, s_m \in \mathcal{F}$ such that:
 - (a) $z_v \in \langle x_{s_1}, \dots, x_{s_m} \rangle$
 - (b) $|s_j| < |v|$, for all $1 \leq j \leq m$.

Remark 5.7. Let us point out that Theorem 5.5 does not seem extendable for arbitrary thin family \mathcal{G} in place of $[\mathbb{N}]^k$. The main difficulty for this is that the elements of \mathcal{G} when $o(\mathcal{G}) \geq \omega$ are not of equal length. Thus in the new thin family $\mathcal{G} \oplus \mathcal{F}$ the plegma pairs are not decomposed into two plegma pairs from the families \mathcal{F} and \mathcal{G} . However a general composition result (i.e. for arbitrary thin family \mathcal{G}) seems provable after a modification of the notion of plegma on $\mathcal{G} \oplus \mathcal{F}$. This is beyond the purposes of the present paper and thus it will be not further discussed.

2. Strong k -order spreading models

In [24] is provided a different notion of k -order spreading model. In this subsection we will recall their definition and we will discuss its relation with the present context. According to [24] we have the following terminology.

- 1) Let E_0, E be Banach spaces. We write $E_0 \rightarrow E$ if E has a Schauder basis which is a spreading model of some seminormalized basic sequence in E_0 and $E_0 \xrightarrow{k} E$ if $E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_{k-1} \rightarrow E$ for some sequence of Banach spaces E_1, \dots, E_{k-1} . Note that for every $k \in \mathbb{N}$ if $E_0 \xrightarrow{k} E$ then E has a subsymmetric Schauder basis.

2) Let E_1, E_2 be Banach spaces with Schauder bases $(e_n^1)_{n \in \mathbb{N}}$ and $(e_n^2)_{n \in \mathbb{N}}$ respectively. We will write $E_1 \xrightarrow{\text{bl}} E_2$ if $(e_n^2)_{n \in \mathbb{N}}$ is a spreading model of some seminormalized block subsequence of $(e_n^1)_{n \in \mathbb{N}}$. Let E_0 be a Banach space and E be a Banach space with a Schauder basis. Similarly for some $k > 1$, we say that $E_0 \xrightarrow{\text{bl}}^k E$, if there exists a sequence E_1, \dots, E_{k-1} of Banach spaces with Schauder bases $(e_n^1)_{n \in \mathbb{N}}, \dots, (e_n^{k-1})_{n \in \mathbb{N}}$ respectively such that $E_0 \xrightarrow{\text{bl}} E_1 \xrightarrow{\text{bl}} E_2 \xrightarrow{\text{bl}} \dots \xrightarrow{\text{bl}} E_{k-1} \xrightarrow{\text{bl}} E$.

Definition 5.8. Let E_0, E be Banach spaces and $k \in \mathbb{N}$. We say that E is a *strong* k -order spreading model of E_0 if $E_0 \xrightarrow{\text{bl}}^k E$. Additionally, E is a *block strong* k -order spreading model of E_0 if $E_0 \xrightarrow{\text{bl}}^k E$.

Under the above definition we have the following which is a direct consequence of Theorem 5.5.

Corollary 5.9. Let X be a Banach space and E be a Banach space with Schauder basis $(e_n)_{n \in \mathbb{N}}$ such that $X \xrightarrow{\text{bl}}^k E$, for some $k > 1$. Then $(e_n)_{n \in \mathbb{N}}$ is a k -order spreading model of X .

Remark 5.10. In [24] is constructed a space which does not contain any ℓ^p , for $1 \leq p \leq \infty$, or c_0 spreading model. In the same paper is asked if there exists a space which does not contain any ℓ^p , for $1 \leq p \leq \infty$, or c_0 strong k -order spreading model of any $k \in \mathbb{N}$. In Chapter 11 we answer affirmatively this problem.

Remark 5.11. Let us also mention that generally the class of block strong k -order spreading models is strictly smaller than the corresponding one consisted by the plegma block generated k -order spreading models. In Chapter 9 we construct a space having this property.

3. Applications to ℓ^p and c_0 spreading models

3.1. ℓ^p spreading models.

Definition 5.12. Let \mathcal{F} be a regular thin family, $M \in [\mathbb{N}]^\infty$, $p \in [1, \infty)$, X be a Banach space and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in X . We say that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F}} \upharpoonright M$ generates ℓ^p (resp. c_0) as an \mathcal{F} -spreading model if $(x_s)_{s \in \mathcal{F}} \upharpoonright M$ generates as an \mathcal{F} -spreading model a sequence equivalent to the usual basis of ℓ^p (resp. c_0).

Remark 5.13. It is easy to see that an \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F}} \upharpoonright M$ generates ℓ^p as an \mathcal{F} -spreading model iff $(x_s)_{s \in \mathcal{F}} \upharpoonright M$ generates an \mathcal{F} -spreading model and there exist $C, c > 0$ such that

$$c \left(\sum_{j=1}^n |a_j|^p \right)^{\frac{1}{p}} \leq \left\| \sum_{j=1}^n a_j x_{s_j} \right\| \leq C \left(\sum_{j=1}^n |a_j|^p \right)^{\frac{1}{p}}$$

for all $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}$ and $(s_j)_{j=1}^n$ plegma n -tuple in $\mathcal{F} \upharpoonright M$ with $s_1(1) \geq M(n)$.

Proposition 5.14. Let X be a Banach space, $\xi < \omega_1$ and $(e_n)_{n \in \mathbb{N}}$ a nontrivial sequence in $\mathcal{SM}_\xi^{wrc}(X)$ such that the space $E = \overline{\langle (e_n)_{n \in \mathbb{N}} \rangle}$ contains an isomorphic copy of ℓ^p (resp. c_0), for some $p \in [1, \infty)$. Then X admits an ℓ^p (resp. c_0) spreading model of order $\xi + 1$.

PROOF. Let \mathcal{F} be a regular thin family of order ξ , $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in X satisfying the conclusion of Lemma 4.6 for $(e_n)_{n \in \mathbb{N}}$. Let $L \in [M]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is subordinated. Let $\widehat{\varphi} : \widehat{\mathcal{F}} \upharpoonright L \rightarrow (X, w)$ be the map witnessing this. Let $x'_s = x_s - \widehat{\varphi}(\emptyset)$ for all $s \in \mathcal{F} \upharpoonright L$. By passing to an infinite subset of L we may suppose that $(x'_s)_{s \in \mathcal{F} \upharpoonright L}$ generates an \mathcal{F} -spreading model $(e'_n)_{n \in \mathbb{N}}$. By Proposition 4.16 the sequence $(e'_n)_{n \in \mathbb{N}}$ is non trivial and by Theorem 3.32 it is also Schauder basic (actually it is unconditional). Since by our assumption the space $E = \langle (e_n)_{n \in \mathbb{N}} \rangle$ contains an isomorphic copy of ℓ^p (or c_0), applying Proposition 4.18 we get a block subsequence $(y_n)_{n \in \mathbb{N}}$ of $(e'_n)_{n \in \mathbb{N}}$ equivalent to the usual basis of ℓ^p (resp. c_0). By passing to a subsequence of $(y_n)_{n \in \mathbb{N}}$ we may suppose that $(y_n)_{n \in \mathbb{N}}$ generates ℓ^p (resp. c_0) as (an order one) spreading model. By Theorem 5.5 (for $k = 1$) the result follows. \square

3.2. Isometric ℓ^1 and c_0 spreading models. The following is one of the well known results due to R. James (c.f. [16]).

Proposition 5.15. Let $(x_n)_{n \in \mathbb{N}}$ be a normalized sequence in a Banach space X equivalent to the usual basis of ℓ^1 (resp. c_0). Then for every $\varepsilon > 0$ there exists a block normalized subsequence $(y_n)_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ which admits lower ℓ^1 constant $1 - \varepsilon$ (resp. lower c_0 constant $1 - \varepsilon$ and upper c_0 constant $1 + \varepsilon$).

Using the standard diagonalization argument the above proposition yields the following.

Proposition 5.16. Let X be a Banach space with a Schauder basis. If X contains an isomorphic copy of ℓ^1 (resp. c_0) then X admits the usual basis of ℓ^1 (resp. c_0) as a block generated spreading model of order one.

The above proposition, Theorem 5.5 and Remark 5.6 readily yield the following.

Corollary 5.17. Let X be a Banach space with a Schauder basis and $(e_n)_{n \in \mathbb{N}}$ be a Schauder basic sequence in $\mathcal{SM}_\xi(X)$, for some $\xi < \omega_1$. Suppose that $\langle (e_n)_{n \in \mathbb{N}} \rangle$ contains an isomorphic copy of ℓ^1 (resp. c_0). Then X admits the usual basis of ℓ^1 (resp. c_0) as a $\xi + 1$ -spreading model.

Precisely, for every regular thin family \mathcal{F} of order ξ , $M \in [\mathbb{N}]^\infty$ and \mathcal{F} -sequence $(x_s)_{s \in \mathcal{F}}$ in X such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model, there exist $L \in [M]^\infty$ and a \mathcal{G} -sequence $(z_v)_{v \in \mathcal{G}}$, where $\mathcal{G} = [\mathbb{N}]^1 \oplus \mathcal{F}$, which satisfy the following:

- (i) The \mathcal{G} -subsequence $(z_v)_{v \in \mathcal{G} \upharpoonright L}$ generates the usual basis of ℓ^1 (resp. c_0) as a \mathcal{G} -spreading model.
- (ii) For every $v \in \mathcal{G} \upharpoonright L$ there exist $m \in \mathbb{N}$ and $s_1, \dots, s_m \in \mathcal{F}$ such that $z_v \in \langle x_{s_1}, \dots, x_{s_m} \rangle$ and $|s_j| < |v|$, for all $1 \leq j \leq m$.

If in addition the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ plegma block generates $(e_n)_{n \in \mathbb{N}}$, then the \mathcal{G} -subsequence $(z_v)_{v \in \mathcal{G} \upharpoonright L}$ plegma block generates the usual basis of ℓ^1 (resp. c_0).

Theorem 5.18. Let X be a Banach space. Then for each $1 \leq \xi < \omega_1$ at least one of the following holds.

- (i) The usual basis of ℓ^1 or c_0 belongs to $\mathcal{SM}_{\xi+1}^{wrc}(X)$.
- (ii) For every $(e_n)_{n \in \mathbb{N}} \in \mathcal{SM}_\xi^{wrc}(X)$, the space $E = \langle (e_n)_{n \in \mathbb{N}} \rangle$ is reflexive.

Moreover every Schauder basic $(e_n)_{n \in \mathbb{N}} \in \mathcal{SM}_\xi^{wrc}(X)$ is unconditional.

PROOF. Let X be a Banach space and $1 \leq \xi < \omega_1$. Assume that the usual basis of ℓ^1 and c_0 do not belong to $\mathcal{SM}_{\xi+1}^{wrc}(X)$. Let $(e_n)_{n \in \mathbb{N}} \in \mathcal{SM}_{\xi}^{wrc}(X)$. We will show that the space $E = \overline{\langle (e_n)_{n \in \mathbb{N}} \rangle}$ is reflexive. Indeed, let \mathcal{F} be a regular thin family of order ξ , $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ satisfying the conclusion of Lemma 4.6. Let $L \in [M]^\infty$ such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is subordinated and let $\widehat{\varphi} : \widehat{\mathcal{F}} \upharpoonright L \rightarrow (X, w)$ be the continuous map witnessing this. Then either $\widehat{\varphi}(\emptyset) = 0$ or $\widehat{\varphi}(\emptyset) \neq 0$.

Let $\widehat{\varphi}(\emptyset) = 0$. Then by Theorem 3.32 we have that $(e_n)_{n \in \mathbb{N}}$ is unconditional. By Corollary 5.17, the space $E = \overline{\langle (e_n)_{n \in \mathbb{N}} \rangle}$ cannot contain any isomorphic copy of c_0 or ℓ^1 . Hence by James theorem (c.f. [16]) we have that E is reflexive.

Let $\widehat{\varphi}(\emptyset) \neq 0$. For every $s \in \mathcal{F} \upharpoonright L$ we set $x'_s = x_s - \widehat{\varphi}(\emptyset)$. Let $N \in [L]^\infty$ such that $(x'_s)_{s \in \mathcal{F} \upharpoonright N}$ generates an \mathcal{F} -spreading model $(e'_n)_{n \in \mathbb{N}}$. By the above the space $E' = \overline{\langle (e'_n)_{n \in \mathbb{N}} \rangle}$ is reflexive. Clearly we may consider that both $(x_s)_{s \in \mathcal{F} \upharpoonright N}$ and $(x'_s)_{s \in \mathcal{F} \upharpoonright N}$ are \mathcal{F} -subsequences in a Banach space with a Schauder basis (for example in $C[0, 1]$). By Lemma 4.17 we have that $E = \overline{\langle (e_n)_{n \in \mathbb{N}} \rangle}$ is isomorphic to a subspace of $\mathbb{R} \oplus E'$. Since E' is reflexive, we have that $\mathbb{R} \oplus E'$ as well as every subspace of it is reflexive. Hence E is reflexive.

Finally by Theorem 4.8 every Schauder basic $(e_n)_{n \in \mathbb{N}} \in \mathcal{SM}_{\xi}^{wrc}(X)$ is unconditional. \square

Corollary 5.19. Let X be a reflexive space. Then one of the following holds.

- (i) The space X admits the usual basis of ℓ^1 as a ξ -order spreading model, for some $\xi < \omega_1$.
- (ii) The space X admits the usual basis of c_0 as a ξ -order spreading model, for some $\xi < \omega_1$.
- (iii) For every nontrivial spreading model $(e_n)_{n \in \mathbb{N}}$ of any order admitted by X , we have that the space $E = \overline{\langle (e_n)_{n \in \mathbb{N}} \rangle}$ is reflexive.

Moreover every Schauder basic $(e_n)_{n \in \mathbb{N}} \in \mathcal{SM}^{wrc}(X)$ is unconditional.

CHAPTER 6

ℓ^1 spreading models

In this chapter we study ℓ^1 spreading models generated by \mathcal{F} -sequences. In the first section we establish the non distortion of ℓ^1 spreading models and thus we extend the corresponding known result for spreading models of order 1 to the arbitrary order. In the second section we present a technique of splitting a generic assignment to an \mathcal{F} -sequence. This technique will be used in the sequel to show that, under certain natural assumptions on the space X , whenever X admits a ℓ^1 spreading model of order ξ then X also admits a plegma block generated one of the same order. Finally in the last section we extend the well known result of H.P. Rosenthal concerning Cesàro summability of weakly null sequence to all spreading models of finite order.

1. Almost isometric ℓ^1 spreading models

Let $M \in [\mathbb{N}]^\infty$, $C \geq c > 0$, \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in a Banach space X . We will say that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates ℓ^1 as an \mathcal{F} -spreading model with constants c, C if $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates an \mathcal{F} -spreading model $(e_n)_n$ which is c, C isomorphic to the standard basis of ℓ^1 , that is

$$c \sum_{j=1}^n |a_j| \leq \left\| \sum_{j=1}^n a_j e_j \right\| \leq C \sum_{j=1}^n |a_j|$$

for all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$. In particular if $\|x_s\| \leq 1$, for all $s \in \mathcal{F} \upharpoonright M$, then we may assume that $C = 1$ and in this case we will say that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates ℓ^1 as an \mathcal{F} -spreading model with constant c .

Proposition 6.1. Let $M \in [\mathbb{N}]^\infty$, $1 > c > 0$, \mathcal{F} be a regular thin family and $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in the unit ball B_X of a Banach space X . Suppose that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates ℓ^1 as an \mathcal{F} -spreading model with constant c . Then for every $\varepsilon > 0$ there exist $L \in [M]^\infty$ and an \mathcal{F} -subsequence $(y_s)_{s \in \mathcal{F} \upharpoonright L}$ in B_X such that the \mathcal{F} -subsequence $(y_s)_{s \in \mathcal{F} \upharpoonright L}$ generates ℓ^1 as an \mathcal{F} -spreading model with constant $1 - \varepsilon$.

PROOF. Let $(e_n)_{n \in \mathbb{N}}$ be the \mathcal{F} -spreading model generated by $(x_s)_{s \in \mathcal{F} \upharpoonright M}$. Then for every $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$, we have that

$$c \sum_{j=1}^n |a_j| \leq \left\| \sum_{j=1}^n a_j e_j \right\| \leq \sum_{j=1}^n |a_j|$$

Clearly we may assume that

$$(10) \quad c = \inf \left\{ \left\| \sum_{j=1}^n a_j e_j \right\| : n \in \mathbb{N} \text{ and } a_1, \dots, a_n \in \mathbb{R} \text{ such that } \sum_{j=1}^n |a_j| = 1 \right\}$$

Let $\varepsilon > 0$ and choose $0 < \varepsilon' < c$ such that

$$\frac{c - \varepsilon'}{c + 2\varepsilon'} > 1 - \varepsilon$$

By passing to an infinite subset of M , if it is necessary, we may assume the following:

- (a) The family \mathcal{F} is very large in M .
- (b) For every $n \in \mathbb{N}$, $a_1, \dots, a_n \in [-1, 1]$ and every plegma n -tuple $(t_j)_{j=1}^n$ in $\mathcal{F} \upharpoonright M$ with $\min t_1 \geq M(n)$,

$$\left\| \sum_{j=1}^n a_j x_{s_j} \right\| - \left\| \sum_{j=1}^n a_j e_j \right\| < \varepsilon'$$

By (10), there exist $k \in \mathbb{N}$ and $b_1, \dots, b_k \in [-1, 1]$ with $\sum_{i=1}^k |b_i| = 1$ such that

$$\left\| \sum_{i=1}^k b_i e_i \right\| < c + \varepsilon'$$

Hence by (b), for every plegma k -tuple $(s_i)_{i=1}^k$ in $\mathcal{F} \upharpoonright M$ with $\min s_1 \geq M(k)$ we have that

$$\left\| \sum_{i=1}^k b_i x_{s_i} \right\| < c + 2\varepsilon'$$

For each $n \in \mathbb{N}$, set $I_n = \{M(n \cdot k + 1), \dots, M((n+1) \cdot k)\}$. Then obviously, $\max(I_n) < \min(I_{n+1})$, $|I_n| = k$ and $\min(I_n) > M(n \cdot k)$. We set

$$L = \{\max I_n : n \in \mathbb{N}\} = \{M((n+1) \cdot k) : n \in \mathbb{N}\}$$

Since $\widehat{\mathcal{F}}$ is regular and \mathcal{F} is very large in M , it is easy to see that for every $1 \leq i \leq k$ there exists a unique $t_i^s \subseteq \{I_{n_j}(i) : 1 \leq j \leq |s|\}$ with $t_i^s \in \mathcal{F}$. We set

$$y_s = \frac{\sum_{j=1}^k b_j x_{t_j^s}}{c + 2\varepsilon'},$$

for all $s \in \mathcal{F} \upharpoonright L$.

We claim that the \mathcal{F} -subsequence $(y_s)_{s \in \mathcal{F} \upharpoonright L}$ generates ℓ^1 as an \mathcal{F} -spreading model with constant $1 - \varepsilon$. Indeed, let $n \in \mathbb{N}$, $a_1, \dots, a_n \in [-1, 1]$ and $(s_j)_{j=1}^n$ plegma n -tuple in $\mathcal{F} \upharpoonright L$ with $s_1(1) \geq L(n)$. First notice that the $n \cdot k$ -tuple $(t_1^{s_1}, \dots, t_k^{s_1}, \dots, t_1^{s_n}, \dots, t_k^{s_n})$ is plegma and $t_1^{s_1}(1) \geq \min I_n > M(n \cdot k)$. Hence

$$\begin{aligned} \left\| \sum_{j=1}^n a_j y_{s_j} \right\| &= \left\| \sum_{j=1}^n a_j \cdot \sum_{i=1}^k \frac{b_i x_{t_i^{s_j}}}{c + 2\varepsilon'} \right\| = \left\| \sum_{j=1}^n \sum_{i=1}^k \frac{a_j}{c + 2\varepsilon'} b_i x_{t_i^{s_j}} \right\| \\ &\geq (c - \varepsilon') \sum_{j=1}^n \sum_{i=1}^k \frac{|a_j| \cdot |b_i|}{c + 2\varepsilon'} = \frac{c - \varepsilon'}{c + 2\varepsilon'} \sum_{j=1}^n |a_j| \sum_{i=1}^k |b_i| = \frac{c - \varepsilon'}{c + 2\varepsilon'} \sum_{j=1}^n |a_j| \end{aligned}$$

□

Remark 6.2. Let us point out that the proof of the above proposition yields that additional properties concerning the initial \mathcal{F} -sequence $(x_s)_{s \in \mathcal{F}}$ pass to the \mathcal{F} -subsequence $(y_s)_{s \in \mathcal{F} \upharpoonright L}$ that generates the almost isometric ℓ^1 \mathcal{F} -spreading model. In particular assuming that X has a Schauder basis and the $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ plegma block (resp. plegma disjointly) generates ℓ^1 as an \mathcal{F} -spreading model then the same holds for $(y_s)_{s \in \mathcal{F} \upharpoonright L}$. Similarly if $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ admits a generic (resp. disjointly generic) decomposition then $(y_s)_{s \in \mathcal{F} \upharpoonright L}$ also does.

2. Splitting the generic assignments

Definition 6.3. Let X be a Banach space with a Schauder basis. Let \mathcal{F} be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $\mathcal{G} \subseteq [M]^{<\infty}$ be a thin family such that $\mathcal{G} \sqsubset \mathcal{F} \upharpoonright M$. Let $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in X such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ admits a generic assignment $(\widehat{\varphi}, (\widetilde{x}_s)_{s \in \mathcal{F} \upharpoonright M}, (\widetilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright M})$.

The \mathcal{G} -splitting of $((x_s)_{s \in \mathcal{F} \upharpoonright M}, \widehat{\varphi}, (\widetilde{x}_s)_{s \in \mathcal{F} \upharpoonright M}, (\widetilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright M})$ is the pair

$$\left(((x_s^{(i, \mathcal{G})})_{s \in \mathcal{F} \upharpoonright M}, \widehat{\varphi}^{(i, \mathcal{G})}, (\widetilde{x}_s^{(i, \mathcal{G})})_{s \in \mathcal{F} \upharpoonright M}, (\widetilde{y}_t^{(i, \mathcal{G})})_{t \in \widehat{\mathcal{F}} \upharpoonright M}) \right)_{i=1}^2$$

which is defined as follows.

(i) (a) For every $t \in \widehat{\mathcal{G}}$,

$$\widetilde{y}_t^{(1, \mathcal{G})} = \widetilde{y}_t \text{ and } \widetilde{y}_t^{(2, \mathcal{G})} = 0$$

(b) For every $t \in (\widehat{\mathcal{F}} \upharpoonright M) \setminus \widehat{\mathcal{G}}$,

$$\widetilde{y}_t^{(1, \mathcal{G})} = 0 \text{ and } \widetilde{y}_t^{(2, \mathcal{G})} = \widetilde{y}_t$$

(ii) (a) For every $t \in \widehat{\mathcal{G}}$,

$$\widehat{\varphi}^{(1, \mathcal{G})}(t) = \widehat{\varphi}(t) \text{ and } \widehat{\varphi}^{(2, \mathcal{G})}(t) = 0$$

(b) For every $t \in (\widehat{\mathcal{F}} \upharpoonright M) \setminus \widehat{\mathcal{G}}$,

$$\widehat{\varphi}^{(1, \mathcal{G})}(t) = \widehat{\varphi}(v_t) \text{ and } \widehat{\varphi}^{(2, \mathcal{G})}(t) = \widehat{\varphi}(v) - \widehat{\varphi}(v_t)$$

where v_t the unique element of \mathcal{G} such that $v_t \sqsubset t$.

(iii) For every $i = 1, 2$ and for all $s \in \mathcal{F} \upharpoonright M$,

$$x_s^{(i, \mathcal{G})} = \widehat{\varphi}^{(i, \mathcal{G})}(s) \text{ and } \widetilde{x}_s^{(i, \mathcal{G})} = \sum_{t \sqsubseteq s} \widetilde{y}_t^{(i, \mathcal{G})}$$

Remark 6.4.

(i) Notice that for each $s \in \mathcal{F} \upharpoonright M$,

$$x_s = x_s^{(1, \mathcal{G})} + x_s^{(2, \mathcal{G})} \text{ and } \widetilde{x}_s = \widetilde{x}_s^{(1, \mathcal{G})} + \widetilde{x}_s^{(2, \mathcal{G})}$$

and for each $t \in \widehat{\mathcal{F}} \upharpoonright M$,

$$\widehat{\varphi}(t) = \widehat{\varphi}^{(1, \mathcal{G})}(t) + \widehat{\varphi}^{(2, \mathcal{G})}(t) \text{ and } \widetilde{y}_t = \widetilde{y}_t^{(1, \mathcal{G})} + \widetilde{y}_t^{(2, \mathcal{G})}$$

(ii) It is easy to see that $(\widehat{\varphi}^{(i, \mathcal{G})}, (\widetilde{x}_s^{(i, \mathcal{G})})_{s \in \mathcal{F} \upharpoonright M}, (\widetilde{y}_t^{(i, \mathcal{G})})_{t \in \widehat{\mathcal{F}} \upharpoonright M})$ is a generic assignment to $(x_s^{(i, \mathcal{G})})_{s \in \mathcal{F} \upharpoonright M}$, for each $i = 1, 2$. Moreover notice that

$$\widehat{\varphi}^{(2, \mathcal{G})}(\emptyset) = 0 \text{ and } \widehat{\varphi}^{(1, \mathcal{G})}(\emptyset) = \widehat{\varphi}(\emptyset)$$

Lemma 6.5. Let X be a Banach space with a Schauder basis. Let \mathcal{F} be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $\mathcal{G} \subseteq [M]^{<\infty}$ such that $\mathcal{G} \sqsubset \mathcal{F} \upharpoonright M$. Suppose that $\mathcal{G}(M^{-1}) = \{s \in [\mathbb{N}]^{<\infty} : M(s) \in \mathcal{G}\}$ is regular thin. Let $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in X such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ admits a generic assignment $(\widehat{\varphi}, (\widetilde{x}_s)_{s \in \mathcal{F} \upharpoonright M}, (\widetilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright M})$. Then every spreading model admitted by $(x_s^{(1, \mathcal{G})})_{s \in \mathcal{F} \upharpoonright M}$ belongs to $\mathcal{SM}_{o(\mathcal{G})}(X)$.

PROOF. Let $L \in [M]^\infty$ such that $(x_s^{(1, \mathcal{G})})_{s \in \mathcal{F} \upharpoonright L}$ generates an \mathcal{F} -spreading model $(e_n)_{n \in \mathbb{N}}$. We set $\mathcal{H} = \mathcal{G}(L^{-1}) = \{s \in [\mathbb{N}]^{<\infty} : L(s) \in \mathcal{G}\}$. By the remarks after Definition 1.21 we have that \mathcal{H} is a regular thin family with $o(\mathcal{H}) = o(\mathcal{G} \upharpoonright L) = o(\mathcal{G})$. For every $t \in \mathcal{H}$ we set $w_t = x_s^{(1, \mathcal{G})}$, where $s \in \mathcal{F} \upharpoonright M$ and $L(t) \sqsubset s$. By the

definition of $(x_s^{(1, \mathcal{G})})_{s \in \mathcal{F} \upharpoonright M}$ we have that w_t is well defined. It is easy to see that $(w_t)_{t \in \mathcal{H}}$ generates $(e_n)_{n \in \mathcal{F}}$ as an \mathcal{H} -spreading model. \square

Definition 6.6. Let \mathcal{F} be regular thin and $L = \{l_1, l_2, \dots\} \in [\mathbb{N}]^\infty$ such that \mathcal{F} is very large in L . We define

$$\mathcal{F}/L = \left\{ \{l_{k_1}, \dots, l_{k_m}\} : m \in \mathbb{N} \text{ and } \{l_{k_1+1}, \dots, l_{k_m+1}\} \in \mathcal{F}_{(l_1)} \upharpoonright L(2\mathbb{N}-1) \right\}$$

(where $L(2\mathbb{N}-1) = \{l_1, l_3, \dots\}$).

The following lemma is easily verified.

Lemma 6.7. Let \mathcal{F} be regular thin and $L \in [\mathbb{N}]^\infty$ such that \mathcal{F} is very large in L . Then the following hold.

- (i) The family \mathcal{F}/L is regular thin.
- (ii) $\mathcal{F}/L \sqsubset \mathcal{F} \upharpoonright L(2\mathbb{N})$.
- (iii) $o(\mathcal{F}/L) = o(\mathcal{F}_{(l_1)}) < o(\mathcal{F})$, where $l_1 = \min L$.

Proposition 6.8. Let X be a Banach space with a Schauder basis and $(e_n)_{n \in \mathbb{N}} \in \mathcal{SM}^{wrc}(X)$. Let

$$\xi_0 = \min \{ \xi < \omega_1 : (e_n)_{n \in \mathbb{N}} \in \mathcal{SM}_\xi^{wrc}(X) \}$$

Let \mathcal{F} be regular thin of order ξ_0 , $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ a relatively weakly compact \mathcal{F} -sequence in X such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model and admits a generic assignment.

Then for every $L \in [M]^\infty$, the \mathcal{F} -subsequence $(x_s^{(1, \mathcal{G})})_{s \in \mathcal{F} \upharpoonright L(2\mathbb{N})}$ does not admit $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model, where $\mathcal{G} = \mathcal{F}/L$.

By the above and Corollary 4.2 we have the following.

Corollary 6.9. Let X be a Banach space with a Schauder basis. Suppose that $\mathcal{SM}^{wrc}(X)$ contains a sequence equivalent to the usual basis of ℓ^1 . Let ξ_0 be the minimum countable ordinal ξ such that $\mathcal{SM}_\xi^{wrc}(X)$ contains a sequence equivalent to the usual basis of ℓ^1 . Let $(e_n)_{n \in \mathbb{N}} \in \mathcal{SM}_{\xi_0}^{wrc}(X)$ equivalent to the usual basis of ℓ^1 and let $c > 0$ be the lower ℓ^1 constant of $(e_n)_{n \in \mathbb{N}}$. Let \mathcal{F} be regular thin of order ξ_0 , $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ a relatively weakly compact \mathcal{F} -sequence in X such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates $(e_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model and a generic assignment.

Then for every $L \in [M]^\infty$ there exists $N \in [L(2\mathbb{N})]^\infty$ such that $(x_s^{(2, \mathcal{G})})_{s \in \mathcal{F} \upharpoonright N}$ generates ℓ^1 as an \mathcal{F} -spreading model with lower constant c , where $\mathcal{G} = \mathcal{F}/L$.

Notation 6.10. Let (s_1, s_2) be a plegma pair in $[\mathbb{N}]^{<\infty}$. We set

$$s_2/s_1 = s_2 \cap \{1, \dots, \max s_1\}$$

Remark 6.11. Let \mathcal{F} be regular thin and $L = \{l_1, l_2, \dots\} \in [\mathbb{N}]^\infty$ such that \mathcal{F} is very large in L . Let us observe that $\mathcal{F}/L \subseteq \{s/s' : (s', s) \in Plm(\mathcal{F} \upharpoonright L)\}$. Precisely, for every $s \in \mathcal{F} \upharpoonright L(2\mathbb{N})$ let $\tilde{s} \in \mathcal{F} \upharpoonright L(2\mathbb{N}-1)$ defined as follows. If $s = \{l_{n_1} < \dots < l_{n_k}\}$, then \tilde{s} is the unique initial segment of $\{l_1, l_{n_1+1}, \dots, l_{n_k-1+1}\}$ in \mathcal{F} . Then it is easy to see that

$$\mathcal{F}/L \subseteq \{s/\tilde{s} : s \in \mathcal{F} \upharpoonright L(2\mathbb{N})\}$$

Lemma 6.12. Let X be a Banach space with a Schauder basis, \mathcal{F} a regular thin family and $M = \{m_1, m_2, \dots\} \in [\mathbb{N}]^\infty$ such that \mathcal{F} is very large in M . Let $(\tilde{x}_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in X , such that $(\tilde{x}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright M}$ admits a generic decomposition $(\tilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright M}$. For every $s \in \mathcal{F} \upharpoonright M(2\mathbb{N})$, with $s = \{M(2\rho_1) < \dots < M(2\rho_k)\}$, we set s' to be the unique initial segment of $\{M(2\rho_1 - 1) < \dots < M(2\rho_k - 1)\}$ in \mathcal{F} and $s^* = s/s'$. For every $s \in \mathcal{F} \upharpoonright M(2\mathbb{N})$, let

$$z_s = \tilde{x}_s - \sum_{t \sqsubseteq s^*} \tilde{y}_t = \sum_{s^* \sqsubset t \sqsubseteq s} \tilde{y}_t$$

Then $(z_s)_{s \in \mathcal{F} \upharpoonright M(2\mathbb{N})}$ is a plegma block \mathcal{F} -subsequence.

PROOF. Let (s_1, s_2) be a plegma pair in $\mathcal{F} \upharpoonright M(2\mathbb{N})$. Then it is easy to see that the (s_1^*, s_1, s_2^*, s_2) is a plegma 4-tuple. Hence $\max s_1 < \max s_2^*$. Therefore $s_1 \setminus (s_1/s_1^*) < s_2 \setminus (s_2/s_2^*)$. Thus if $\max((s_1/s_1^*)) = s_1(k)$ and $\min(s_2/s_2^*) = s_2(l)$, then we have that $k \leq l$. By the definition of the generic decomposition we have that $z_{s_1} < z_{s_2}$. \square

3. Plegma block generated ℓ^1 spreading models

Definition 6.13. Let X be a Banach space with a Schauder basis $(e_n)_{n \in \mathbb{N}}$. We say that X satisfies the property \mathcal{P} , if for every $\delta > 0$ there exists $k \in \mathbb{N}$ such that for every finite block sequence $(x_j)_{j=1}^k$, with $\|x_j\| \geq \delta$ for all $j = 1, \dots, k$, we have that $\|\sum_{j=1}^k x_j\| > 1$.

Remark 6.14.

- (i) It is easy to see that the property \mathcal{P} is preserved under equivalent renormings.
- (ii) If X is a Banach space with an unconditional basis $(e_n)_{n \in \mathbb{N}}$ then X satisfies the property \mathcal{P} iff c_0 is not finitely block representable in X .

Theorem 6.15. Let X be a Banach space with a Schauder basis $(e_n)_{n \in \mathbb{N}}$ satisfying property \mathcal{P} . Let $\xi < \omega_1$ and \mathcal{F} be a regular thin family of order ξ . Let $(x_s)_{s \in \mathcal{F}}$ be a weakly relatively compact \mathcal{F} -sequence in X and $M \in [\mathbb{N}]^\infty$ such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates ℓ^1 as an \mathcal{F} -spreading model. Then there exist an \mathcal{F} -sequence $(z_s)_{s \in \mathcal{F}}$ in X and $L \in [M]^\infty$ such that ℓ^1 is plegma block generated as an \mathcal{F} -spreading model by the \mathcal{F} -subsequence $(z_s)_{s \in \mathcal{F} \upharpoonright L}$.

Therefore, if for some $\xi < \omega_1$ ℓ^1 is admitted as a ξ -order spreading model by a weakly relatively compact subset of X , then ℓ^1 is plegma block generated as a ξ -order spreading model.

Lemma 6.16. Let \mathcal{F} be regular thin, $M \in [\mathbb{N}]^\infty$, $0 < \delta < c \leq 1$ and X be a Banach space with a basis. Let $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in B_X , such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ admits a generic assignment $(\widehat{\varphi}, (\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright M}, (\tilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright M})$. Assume that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates ℓ^1 as an \mathcal{F} -spreading model with constant c . If for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright M$ we have that $\|\sum_{t \sqsubseteq s_2/s_1} \tilde{y}_t\| < \delta$, then X admits ℓ^1 as a plegma block generated \mathcal{F} -spreading model.

PROOF. We may suppose that \mathcal{F} is very large in M . Let $(z_s)_{s \in \mathcal{F} \upharpoonright M(2\mathbb{N})}$ be the \mathcal{F} -subsequence defined in Lemma 6.12. Then for every $s \in \mathcal{F} \upharpoonright M(2\mathbb{N})$,

$$\|\tilde{x}_s - z_s\| = \left\| \sum_{t \sqsubseteq s^*} \tilde{y}_t \right\| < \delta$$

This implies that every \mathcal{F} -spreading model admitted by $(z_s)_{s \in \mathcal{F} \upharpoonright M(2\mathbb{N})}$ is equivalent to the usual basis of ℓ_1 (with lower constant $c - \delta$). Hence ℓ_1 is plegma block generated by the \mathcal{F} -subsequence $(z_s)_{s \in \mathcal{F} \upharpoonright M(2\mathbb{N})}$ as an \mathcal{F} -spreading model. \square

PROOF OF THEOREM 6.15. By Remark 2.4 and Remark 6.14 we may suppose that the basis of X is bimonotone. Let $\xi < \omega_1$ be the minimum countable ordinal such that there exists a weakly relatively compact subset A of X such that ℓ^1 is isomorphic to a ξ -order spreading model of A .

Let \mathcal{F} be regular thin of order ξ , $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ a relatively weakly compact \mathcal{F} -sequence in B_X such that the subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates ℓ^1 as an \mathcal{F} -spreading model with constant c . By Proposition 4.14 there exists $M_1 \in [M]^\infty$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright M_1}$ admits a generic assignment $(\widehat{\varphi}, (\widetilde{x}_s)_{s \in \mathcal{F} \upharpoonright M_1}, (\widetilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright M_1})$.

Let $0 < \delta < c$ and let $k \in \mathbb{N}$ such that for every finite block sequence $(x_j)_{j=1}^k$, with $\|x_j\| \geq \delta$ for all $j = 1, \dots, k$, $\|\sum_{j=1}^k x_j\| > 1$. By Proposition 1.12 there exists $M'_1 \in [M_1]^\infty$ satisfying one of the following:

- (i) $\|\sum_{t \sqsubseteq s_2/s_1} \widetilde{y}_t\| \geq \delta$ for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright M'_1$,
- (ii) $\|\sum_{t \sqsubseteq s_2/s_1} \widetilde{y}_t\| < \delta$ for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright M'_1$.

If (ii) occurs then the result follows by Lemma 6.16. Otherwise by Lemma 6.9 there exists $M_2 \in [M'_1(2\mathbb{N})]^\infty$ such that $(x_s^{(2, \mathcal{G}_1)})_{s \in \mathcal{F} \upharpoonright M_2}$ generates ℓ^1 as an \mathcal{F} -spreading model with lower constant c , where $\mathcal{G}_1 = \mathcal{F}/M'_1$.

We set $x_s^2 = x_s^{(2, \mathcal{G}_1)}$, $\widetilde{x}_s^2 = \widetilde{x}_s^{(2, \mathcal{G}_1)}$, for all $s \in \mathcal{F} \upharpoonright M_2$ and $\widehat{\varphi}^2(t) = \widehat{\varphi}^{(2, \mathcal{G}_1)}(t)$, $\widetilde{y}_t^2 = \widetilde{y}_t^{(2, \mathcal{G}_1)}$, for all $t \in \widehat{\mathcal{F}} \upharpoonright M_2$. Similarly by Proposition 1.12 there exists $M'_2 \in [M_2]^\infty$ satisfying one of the following:

- (i) $\|\sum_{t \sqsubseteq s_2/s_1} \widetilde{y}_t^2\| \geq \delta$ for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright M'_2$,
- (ii) $\|\sum_{t \sqsubseteq s_2/s_1} \widetilde{y}_t^2\| < \delta$ for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright M'_2$.

Again if (ii) occurs then the result follows by Lemma 6.16. Otherwise by Lemma 6.9 there exists $M_3 \in [M'_2(2\mathbb{N})]^\infty$ such that $(x_s^{(2, \mathcal{G}_2)})_{s \in \mathcal{F} \upharpoonright M_3}$ generates ℓ^1 as an \mathcal{F} -spreading model with lower constant c , where $\mathcal{G}_2 = \mathcal{F}/M'_2$.

We set $x_s^3 = x_s^{(2, \mathcal{G}_2)}$, $\widetilde{x}_s^3 = \widetilde{x}_s^{(2, \mathcal{G}_2)}$, for all $s \in \mathcal{F} \upharpoonright M_3$ and $\widehat{\varphi}^3(t) = \widehat{\varphi}^{(2, \mathcal{G}_2)}(t)$, $\widetilde{y}_t^3 = \widetilde{y}_t^{(2, \mathcal{G}_2)}$, for all $t \in \widehat{\mathcal{F}} \upharpoonright M_3$. Notice that $\widetilde{x}_s^{(1, \mathcal{G}_1)} < \widetilde{x}_s^{(2, \mathcal{G}_2)}$, $\|\widetilde{x}_s^{(1, \mathcal{G}_1)}\| \geq \delta$, $\|\widetilde{x}_s^{(2, \mathcal{G}_2)}\| \geq \delta$ and $\|\widetilde{x}_s^{(1, \mathcal{G}_1)} + \widetilde{x}_s^{(2, \mathcal{G}_2)}\| \leq 1$. Continuing in the same way it is clear by the property \mathcal{P} of the space X that after at most k steps case (ii) will occur. \square

Corollary 6.17. Let X be a reflexive Banach space with a Schauder basis $(e_n)_{n \in \mathbb{N}}$ satisfying property \mathcal{P} . Let $\xi < \omega_1$ and \mathcal{F} be a regular thin family of order ξ . Let $(x_s)_{s \in \mathcal{F}}$ be an \mathcal{F} -sequence in X and $M \in [\mathbb{N}]^\infty$ such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates ℓ^1 as an \mathcal{F} -spreading model. Then there exist an \mathcal{F} -sequence $(z_s)_{s \in \mathcal{F}}$ in X and $L \in [M]^\infty$ such that ℓ^1 is plegma block generated as an \mathcal{F} -spreading model by the \mathcal{F} -subsequence $(z_s)_{s \in \mathcal{F} \upharpoonright L}$.

Therefore, if for some $\xi < \omega_1$ ℓ^1 is admitted as a ξ -order spreading model, then ℓ^1 is plegma block generated as a ξ -order spreading model.

Remark 6.18. As we show in Chapter 10, the assumption that the space X satisfies property \mathcal{P} is necessary in Theorem 6.15.

4. k -Cesàro summability vs k -order ℓ^1 spreading models

In this section we focus on spreading models of k -order, with $k \in \mathbb{N}$, and especially we study the relation between the k -Cesàro summability of an $[\mathbb{N}]^k$ -sequence and the spreading model that it generates. The main result of this section is Theorem 6.28.

Definition 6.19. Let X be a Banach space, $x_0 \in X$, $k \in \mathbb{N}$, $(x_s)_{s \in [\mathbb{N}]^k}$ be a $[\mathbb{N}]^k$ -sequence in X and $M \in [\mathbb{N}]^\infty$. We will say that the $[\mathbb{N}]^k$ -subsequence $(x_s)_{s \in [M]^k}$ is k -Cesàro summable to x_0 if

$$\binom{n}{k}^{-1} \sum_{s \in [M|n]^k} x_s \xrightarrow[n \rightarrow \infty]{\|\cdot\|} x_0$$

where $M|n = \{M(1), \dots, M(n)\}$.

Remark 6.20. Let X be a Banach space, $x_0 \in X$, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a $[\mathbb{N}]^k$ -sequence in X . It is easy to see that if the $[\mathbb{N}]^k$ -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ norm converges to x_0 , then $(x_s)_{s \in [\mathbb{N}]^k}$ is k -Cesàro summable to x_0 . Moreover if $(x_s)_{s \in [\mathbb{N}]^k}$ is weakly convergent and in addition is k -Cesàro summable to x_0 , then x_0 is the weak limit of $(x_s)_{s \in [\mathbb{N}]^k}$.

Proposition 6.21. Let X be a Banach space with separable dual, $x_0 \in X$, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a $[\mathbb{N}]^k$ -sequence in X . Assume that for every $M \in [\mathbb{N}]^\infty$, $(x_s)_{s \in [M]^k}$ is k -Cesàro summable to x_0 . Then there exists $L \in [\mathbb{N}]^\infty$ such that $(x_s)_{s \in [L]^k}$ weakly converges to x_0 .

PROOF. First observe that for every $x^* \in X^*$, $\varepsilon > 0$ and $M \in [\mathbb{N}]^\infty$ there exists an $L \in [M]^\infty$ such that $|x^*(x_s) - x^*(x_0)| < \varepsilon$ for all $s \in [L]^k$. Next for a norm dense subset $\{x_n^* : n \in \mathbb{N}\}$ of X^* we inductively choose an $L \in [\mathbb{N}]^\infty$ such that for every $n \in \mathbb{N}$ and $s \in [L]^k$ with $\min s \geq L(n)$ we have that $|x_n^*(x_s) - x_n^*(x_0)| < \frac{1}{n}$ for all $1 \leq i \leq n$. This yields that $(x_s)_{s \in [L]^k}$ weakly converges to x_0 . \square

Remark 6.22. It remains open if the above result remains valid without any restriction in X^* .

We will need a deep density Ramsey theorem of H. Furstenberg and Y. Katznelson (c.f. [8]). Actually we will use a reformulation of this result which is due to W. T. Gowers (c.f. [12]) and it is stated as follows.

Theorem 6.23. Let $\delta > 0$, $k \in \mathbb{N}$ and F be a finite subset of \mathbb{Z}^k . Then there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, every subset A of $\{1, \dots, N\}^k$ of size at least δN^k has a subset of the form $a + dF$ for some $a \in \mathbb{Z}^k$ and $d \in \mathbb{N}$.

Lemma 6.24. Let $\delta > 0$ and $k, l \in \mathbb{N}$. Then there exists $N_0 \in \mathbb{N}$ such that for every $N \geq N_0$ and every subset A of the set $[\{1, \dots, N\}]^k$ of all k -subsets of $\{1, \dots, N\}$ of size at least $\delta \binom{N}{k}$, there exists a plegma l -tuple $(s_j)_{j=1}^l$ in A .

PROOF. For every $1 \leq j \leq l$, let $s_j = (j, l+j, 2l+j, \dots, (k-1)l+j)$. Clearly $(s_j)_{j=1}^l$ is plegma l -tuple in $[\mathbb{N}]^k$. We also set $t = (1, \dots, k)$ and let

$$F = \{t, s_1, \dots, s_l\}$$

Since

$$\lim_{n \rightarrow \infty} \frac{\binom{n}{k}}{n^k} = \frac{1}{k!}$$

there exists $N'_0 \in \mathbb{N}$ such that for every $N \geq N'_0$ and every subset A of $[\{1, \dots, N\}]^k$ of size at least $\delta(\frac{N}{k})$ has density at least $\frac{\delta}{2k!}$ in $\{1, \dots, N\}^k$. Hence Theorem 6.23 (applied for $\frac{\delta}{2k!}$ in place of δ) yields that there exists $N_0 \geq N'_0$ such that for every $N \geq N_0$, every subset A of $[\{1, \dots, N\}]^k$ of size at least $\delta(\frac{N}{k})$ has a subset of the form $a + dF$ for some $a \in \mathbb{Z}^k$ and $d \in \mathbb{N}$. We will complete the proof by showing that $(a + ds_j)_{j=1}^l$ is plegma.

Indeed, since $d > 0$, $a(i) + ds_{j_1}(i) < a(i) + ds_{j_2}(i)$, for all $1 \leq i \leq k$ and $1 \leq j_1 < j_2 \leq l$. Hence (since $s_j \in [\mathbb{N}]^k$, for all $1 \leq j \leq l$), it suffices to show that for every $1 \leq i \leq k-1$,

$$a(i) + ds_l(i) < a(i+1) + ds_1(i+1)$$

Indeed, fix such an i . Since $a + dt \in A \subseteq [\{1, \dots, N\}]^k$, we get that for every $a(i) + dt(i) < a(i+1) + dt(i+1)$ or $a(i+1) + d > a(i)$. Therefore,

$$a(i) + ds_l(i) = a(i) + d(s_l(i+1) - 1) < a(i+1) + d + ds_1(i+1) - d = a(i+1) + ds_1(i+1)$$

□

Remark 6.25. It is easy to see that for $k = 1$ the preceding lemma trivially holds (it suffices to set $N_0 = \lceil \frac{l}{\delta} \rceil$) and therefore Theorem 6.23 is actually used for $k \geq 2$. However it is not completely clear to us if the full strength of such a deep theorem like Furstenberg-Katznelson's is actually necessary for the proof of Lemma 6.24.

Lemma 6.26. Let X be a Banach space, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a bounded $[\mathbb{N}]^k$ -sequence in X . Let $M \in [\mathbb{N}]^\infty$ such that the subsequence $(x_s)_{s \in [M]^k}$ generates a Cesàro summable to zero $[\mathbb{N}]^k$ -spreading model $(e_n)_{n \in \mathbb{N}}$. Then for every $L \in [M]^\infty$ the subsequence $(x_s)_{s \in [L]^k}$ is k -Cezaro summable to zero.

PROOF. Assume on the contrary that there exists $L \in [M]^\infty$ such that

$$\lim_n \binom{n}{k}^{-1} \left\| \sum_{s \in [L|n]^k} x_s \right\| \neq 0$$

Then there exists a $\theta > 0$ and a strictly increasing sequence $(p_n)_n$ of natural numbers such that for every $n \in \mathbb{N}$,

$$\binom{p_n}{k}^{-1} \left\| \sum_{s \in [L|p_n]^k} x_s \right\| > \theta$$

Hence there exists a $x_n^* \in S_{X^*}$ such that

$$\binom{p_n}{k}^{-1} x_n^* \left(\sum_{s \in [L|p_n]^k} x_s \right) > \theta$$

For each $n \in \mathbb{N}$, let

$$A_n = \left\{ t \in [\{1, \dots, p_n\}]^k : x_n^*(x_{L(t)}) > \frac{\theta}{2} \right\},$$

where $K = \sup\{\|x_s\| : s \in [\mathbb{N}]^k\}$ and $L(t) = \{L(t(1)), \dots, L(t(|t|))\}$. Then

$$\begin{aligned} \theta \binom{p_n}{k} &< \sum_{s \in [L|p_n]^k} x_n^*(x_s) \\ &= \sum_{t \in A_n} x_n^*(x_{L(t)}) + \sum_{t \in [\{1, \dots, p_n\}]^k \setminus A_n} x_n^*(x_{L(t)}) \\ &\leq |A_n|K + \frac{\theta}{2} \left(\binom{p_n}{k} - |A_n| \right) \leq |A_n|K + \frac{\theta}{2} \binom{p_n}{k} \end{aligned}$$

The latter yields that

$$|A_n| \geq \frac{\theta}{2K} \binom{p_n}{k}$$

For every $m \in \mathbb{N}$ applying Lemma 6.24 for $\delta = \frac{\theta}{2K}$ and $l = 2m - 1$ there exists N_m such that for every $N \geq N_m$ there exists a plegma $(2m - 1)$ -tuple $(t_j)_{j=1}^{2m-1}$ in A_N . Notice that the m -tuple $(t_j)_{j=m}^{2m-1}$ is plegma with $\min t_m \geq m$ and $x_N^*(x_{L(t_j)}) > \frac{\theta}{2}$, for all $m \leq j \leq 2m - 1$. Setting $s_i = L(t_{m-1+i})$ for all $1 \leq i \leq m$, we have that $(s_i)_{i=1}^m$ is a plegma m -tuple in $[L]^k$ with $\min s_1 \geq L(m)$ and

$$\left\| \frac{1}{m} \sum_{j=1}^m x_{s_j} \right\| > \frac{\theta}{2}$$

This easily contradicts that $(e_n)_{n \in \mathbb{N}}$ is Cesàro summable to zero. \square

The following result is immediate by Proposition 4.3 and Lemma 6.26.

Proposition 6.27. Let X be a Banach space, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a bounded $[\mathbb{N}]^k$ -sequence in X . Let $M \in [\mathbb{N}]^\infty$ such that the subsequence $(x_s)_{s \in [M]^k}$ generates an unconditional $[\mathbb{N}]^k$ -spreading model $(e_n)_{n \in \mathbb{N}}$. Then at least one of the following holds:

- (i) The sequence $(e_n)_{n \in \mathbb{N}}$ is equivalent to the standard basis of ℓ^1 .
- (ii) For every $L \in [M]^\infty$ the subsequence $(x_s)_{s \in [L]^k}$ is k -Cesàro summable to zero.

Theorem 6.28. Let X be a Banach space, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a weakly relatively compact $[\mathbb{N}]^k$ -sequence in X . Then there exists $M \in [\mathbb{N}]^\infty$ such that at least one of the following holds:

- (i) The subsequence $(x_s)_{s \in [M]^k}$ generates an $[\mathbb{N}]^k$ -spreading model equivalent to the standard basis of ℓ^1 .
- (ii) There exists $x_0 \in X$ such that for every $L \in [M]^\infty$ the subsequence $(x_s)_{s \in [L]^k}$ is k -Cesàro summable to x_0 .

PROOF. Let $M_1 \in [\mathbb{N}]^\infty$ such that the subsequence $(x_s)_{s \in [M_1]^k}$ generates an $[\mathbb{N}]^k$ -spreading model $(e_n)_{n \in \mathbb{N}}$. By Proposition 3.11 there exists $M_2 \in [M_1]^\infty$ such that the subsequence $(x_s)_{s \in [M_2]^k}$ is subordinated. Let $\widehat{\varphi} : [M_2]^k \rightarrow (X, w)$ be the continuous map witnessing this. Let $x_0 = \widehat{\varphi}(\emptyset)$. For every $s \in [M_2]^k$ we set $x'_s = x_s - \widehat{\varphi}(\emptyset)$. Notice that the map $\widehat{\psi} : [M_2]^k \rightarrow (X, w)$ defined by $\widehat{\psi}(t) = \widehat{\varphi}(t) - \widehat{\varphi}(\emptyset)$ is continuous. Hence $(x'_s)_{s \in [M_2]^k}$ is subordinated. Observe that $\widehat{\psi}(\emptyset) = 0$. Let $M \in [M_2]^\infty$ such that the \mathcal{F} -subsequence $(x'_s)_{s \in \mathcal{F}|L_2}$ generates $(e'_n)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model. By Theorem 3.32 we have that $(e'_n)_{n \in \mathbb{N}}$ is unconditional. By Proposition 6.27 we have that either $(e'_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^1 or for every $L \in [M]^\infty$ the subsequence $(x'_s)_{s \in [L]^k}$ is k -Cesàro summable to zero.

The first alternative using Corollary 4.2 yields that $(e_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^1 . The second alternative, as $x_s = x'_s + x_0$, easily yields that for every $L \in [M]^\infty$ the subsequence $(x_s)_{s \in [L]^k}$ is k -Cesàro summable to x_0 . \square

Remark 6.29. Let us point out here that the case $k = 1$ of Proposition 6.27 is actually a well known theorem due to H. Rosenthal. Notice also that in the case $k = 1$ the two alternatives are mutually exclusive. This does not remain valid for $k \geq 2$. Indeed, in Example 1 setting $\mathcal{F}_\xi = [\mathbb{N}]^k$ we have that the basis $(e_s)_{s \in [\mathbb{N}]^k}$ generates an ℓ^1 spreading model of order k . Moreover for every $L \in [\mathbb{N}]^\infty$ the subsequence $(e_s)_{s \in [L]^k}$ is k -Cesàro summable to zero. To see this let $L \in [\mathbb{N}]^\infty$ and $n \in \mathbb{N}$. Then every plegma tuple in $[L|n]^k$ is of size less than n . Thus

$$\|(\binom{n}{k})^{-1} \sum \{x_s : s \in [L|n]^k\}\| \leq n(\binom{n}{k})^{-1} \rightarrow 0.$$

Let us close this section by giving the following definition which is an attempt to introduce a transfinite Cesàro summability.

Definition 6.30. Let \mathcal{F} be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $n \in \mathbb{N}$. We set $\mathcal{F} \upharpoonright (M|n) = \{s \in \mathcal{F} : s \subseteq M|n\}$. An \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ in Banach space X is said to be \mathcal{F} -Cesàro summable to $x_0 \in X$ if

$$\frac{1}{|\mathcal{F} \upharpoonright (M|n)|} \sum_{s \in \mathcal{F} \upharpoonright (M|n)} x_s \xrightarrow{\|\cdot\|} x_0$$

The following problem is the transfinite analogue of Theorem 6.28.

Problem 3. Let \mathcal{F} be a regular thin family, X be a Banach space and $(x_s)_{s \in \mathcal{F}}$ be a weakly relatively compact \mathcal{F} -sequence in X . Does there exist $M \in [\mathbb{N}]^\infty$ such that at least one of the following holds:

- (i) The subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates an \mathcal{F} -spreading model equivalent to the standard basis of ℓ^1 .
- (ii) There exists $x_0 \in X$ such that for every $L \in [M]^\infty$ the subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ is \mathcal{F} -Cesàro summable to x_0 .

CHAPTER 7

c_0 spreading models

In this chapter we study c_0 spreading models generated by \mathcal{F} -sequences. In the first section we present a combinatorial result concerning partial unconditionality in infinitely branching trees in Banach spaces. Based on this result and using the splitting technique of generic assignments, presented in the previous chapter, we establish a corresponding to ℓ^1 result for plegma block generated c_0 spreading models. Finally in the last section we deal with the duality between c_0 and ℓ^1 spreading models.

1. On partial unconditionality of trees in Banach spaces

In this section we present a Ramsey result concerning partial unconditionality in trees in Banach spaces. Our approach is related to the corresponding one, stated for sequences instead of trees, and which is followed by several papers (see [2], [3], [7], [23]).

We start with some definitions.

Definition 7.1. Let $k \in \mathbb{N}$ and $\Delta = (N^{(1)}, \dots, N^{(k)})$ be a partition of \mathbb{N} into k infinite disjoint sets, i.e. $N^{(i)} \in [\mathbb{N}]^\infty$ for all $1 \leq i \leq k$, $N^{(i)} \cap N^{(j)} = \emptyset$ for all $i \neq j$ in $\{1, \dots, k\}$ and $\mathbb{N} = \bigcup_{i=1}^k N^{(i)}$. For every $L \in [\mathbb{N}]^\infty$ we define the modulo Δ partition of L as the k -tuple $(L_{(\Delta,1)}, \dots, L_{(\Delta,k)})$, where $L_{(\Delta,i)} = L \cap N^{(i)}$ for all $1 \leq i \leq k$. Moreover for every nonempty $F \in [\mathbb{N}]^{<\infty}$ we define the modulo Δ partition of F as the k -tuple $(F_{(\Delta,1)}, \dots, F_{(\Delta,k)})$, where $F_{(\Delta,i)} = F \cap N^{(i)}$ for all $1 \leq i \leq k$, and for $F = \emptyset$, $\emptyset_{\Delta,i} = \emptyset$, for all $1 \leq i \leq k$.

Finally we define the map $i_\Delta : \mathbb{N} \rightarrow \{1, \dots, k\}$ such that $i_\Delta(n) = i$ if $n \in N^{(i)}$, for all $n \in \mathbb{N}$.

Remark 7.2. It is immediate that the following are satisfied:

- ($\Delta 1$) For every $L \in [\mathbb{N}]^\infty$ we have that $L_{(\Delta,i)} = \{L(n) : i_\Delta(n) = i\}$, for all $1 \leq i \leq k$. Moreover $\bigcup_{i=1}^k L_{(\Delta,i)} = L$ and $L_{(\Delta,i)} \cap L_{(\Delta,j)} = \emptyset$, for all $i \neq j$.
- ($\Delta 2$) For every $F \in [\mathbb{N}]^{<\infty}$ and every $L \in [\mathbb{N}]^\infty$ with $F \subseteq L$ we have that

$$F_{(\Delta,i)} = L_{(\Delta,i)} \cap F \subseteq L_{(\Delta,i)}$$

for all $1 \leq i \leq k$. Therefore for every $F \in [\mathbb{N}]^{<\infty}$ and every $L, L' \in [\mathbb{N}]^\infty$ with $F \subseteq L$ and $F \subseteq L'$, we have that

$$L_{(\Delta,i)} \cap F = L'_{(\Delta,i)} \cap F = F_{(\Delta,i)}$$

for all $1 \leq i \leq k$.

- ($\Delta 3$) Let $L, L' \in [\mathbb{N}]^\infty$ and $n_0 \in \mathbb{N}$ such that for every $n \neq n_0$ in \mathbb{N} , $L(n) = L'(n)$. Then

- (a) For every $i \neq i_\Delta(n_0)$ in $\{1, \dots, k\}$, $L_{(\Delta,i)} = L'_{(\Delta,i)}$,
- (b) $L(n_0) \in L_{(\Delta,i_\Delta(n_0))}$ and $L'(n_0) \in L'_{(\Delta,i_\Delta(n_0))}$,

$$(c) \ L_{(\Delta, i_{\Delta}(n_0))} \setminus \{L(n_0)\} = L'_{(\Delta, i_{\Delta}(n_0))} \setminus \{L'(n_0)\}.$$

Definition 7.3. Let $(y_t)_{t \in [\mathbb{N}]^{<\infty}}$ be a family of vectors in a Banach space X . We will say that $(y_t)_{t \in [\mathbb{N}]^{<\infty}}$ is a weakly null tree for every $t \in [\mathbb{N}]^{<\infty}$, setting $N_t = \{n \in \mathbb{N} : n > t\}$, we have that $(y_{t \cup \{n\}})_{n \in N_t} \xrightarrow{w} 0$. A weakly null tree $(y_t)_{t \in [\mathbb{N}]^{<\infty}}$ will be called bounded if the family $(x_s)_{s \in [\mathbb{N}]^{<\infty}}$ is bounded, where $x_s = \sum_{t \sqsubseteq s} y_t$ for all $s \in [\mathbb{N}]^{<\infty}$.

Notation 7.4. Let $N \in [\mathbb{N}]^\infty$ and \mathcal{G} a thin family very large in N . For every $L \in [N]^\infty$ we set $I_{\mathcal{G}}(L)$ to be the unique initial segment of L in \mathcal{G} .

Definition 7.5. Let $k \in \mathbb{N}$, $\Delta = (N^{(1)}, \dots, N^{(k)})$ be a partition of \mathbb{N} into k infinite disjoint sets, $N \in [\mathbb{N}]^\infty$, $\mathfrak{G} = (\mathcal{G}_1, \dots, \mathcal{G}_k)$ be a k -tuple of thin families which are very large in N and $(y_t)_{t \in [\mathbb{N}]^{<\infty}}$ be a bounded weakly null tree in a Banach space X . Also let $\varepsilon > 0$ and $(\delta_n)_{n \in \mathbb{N}}$ be a sequence of positive reals.

We will say that an infinite subset L of N is $(\varepsilon, (\delta_n)_{n \in \mathbb{N}})$ -good (with respect to $k, \Delta, N, \mathfrak{G}$ and $(y_t)_{t \in [\mathbb{N}]^{<\infty}}$) if for every $x^* \in B_{X^*}$ there exists $y^* \in B_{X^*}$ such that, setting for $1 \leq i \leq k$, $v_i = I_{\mathcal{G}_i}(L_{(\Delta, i)})$, the following are satisfied.

- (a) $\left| (x^* - y^*) \left(\sum_{t \sqsubseteq v_i} y_t \right) \right| \leq \varepsilon$.
- (b) For all $t \in [L]^{<\infty}$ with $v_i \sqsubset t \sqsubseteq L_{(\Delta, i)}$, for some $i \in \{1, \dots, k\}$, we have that $|y^*(y_t)| \leq \delta_{|t|}$.

The main goal of this section is to prove the next result.

Theorem 7.6. Let $k, \Delta, N, \mathfrak{G}$ and $(y_t)_{t \in [\mathbb{N}]^{<\infty}}$ be as in Definition 7.5. Also let $\varepsilon > 0$ and $(\delta_n)_{n \in \mathbb{N}}$ be a sequence of positive reals. Then there exists $M \in [N]^\infty$ such that every $L \in [M]^\infty$ is $(\varepsilon, (\delta_n)_{n \in \mathbb{N}})$ -good.

We fix for the sequel $k, \Delta, N, \mathfrak{G}$ and $(y_t)_{t \in [\mathbb{N}]^{<\infty}}$ as in Definition 7.5, $\varepsilon > 0$ and $(\delta_n)_{n \in \mathbb{N}}$ a sequence of positive reals. Let

$$\mathbb{G} = \{L \in [N]^\infty : L \text{ is } (\varepsilon, (\delta_n)_{n \in \mathbb{N}})\text{-good}\}$$

The proof of the following lemma is easy.

Lemma 7.7. The set \mathbb{G} is closed in $[N]^\infty$.

Lemma 7.8. The set \mathbb{G} is dense in $[N]^\infty$, i.e. for each $M \in [N]^\infty$ there exists $L \in \mathbb{G}$ with $L \in [M]^\infty$.

Lemma 7.7, Lemma 7.8 and Galvin-Prikry's theorem yield Theorem 7.6. Therefore it remains to show Lemma 7.8.

To this end we will need the next definition.

Definition 7.9. An $F \in [N]^{<\infty}$ will be called \mathfrak{G} -free if for $i = i_{\Delta}(|F| + 1)$ we have that $F_{(\Delta, i)} \notin \widehat{\mathcal{G}}_i \setminus \mathcal{G}_i$.

Remark 7.10. Notice that by Def. 7.1, we have that for every $F \in [N]^{<\infty}$ and every $n \in N$ with $F < n$, setting $F' = F \cup \{n\}$,

$$F'_{(\Delta, i_{\Delta}(|F'|))} = F_{(\Delta, i_{\Delta}(|F|+1))} \cup \{n\}$$

Hence if $F \in [N]^{<\infty}$ is \mathfrak{G} -free then $F'_{(\Delta, i_{\Delta}(|F'|))} \notin \widehat{\mathcal{G}}_{i_{\Delta}(|F'|)}$.

Let for every $s \in [\mathbb{N}]^{<\infty}$, $x_s = \sum_{t \sqsubseteq s} y_t$ and $K = \sup\{\|x_s\| : s \in [\mathbb{N}]^{<\infty}\}$. Let also $\delta_0 > 0$.

Sublemma 7.11. Let $F \in [\mathbb{N}]^{<\infty}$ be \mathfrak{G} -free and $M \in [N]^\infty$ with $F \sqsubseteq M$. Also let $((a_t)_{t \sqsubseteq F_{(\Delta, i)}})_{i=1}^k$ and $(b_i)_{i=1, i \neq i_0}^k$ in $[-2K, 2K]$, where $i_0 = i_\Delta(|F| + 1)$. For every $L' \in [M]^\infty$ we set $B^*(L')$ to be the set of all $x^* \in B_{X^*}$ satisfying the following.

- (i) For all $1 \leq i \leq k$ and $t \sqsubseteq F_{(\Delta, i)}$, $|x^*(y_t) - a_t| \leq \frac{\delta_{|t|}}{2^{|F|+3}}$.
- (ii) For all $1 \leq i \leq k$ with $i \neq i_0$, $|x^*(\sum_{F_{(\Delta, i)} \sqsubset t \sqsubseteq v'_i} y_t) - b_i| \leq \frac{\varepsilon}{2^{|F|+3}}$, where $v'_i = I_{\mathcal{G}_i}(L'_{(\Delta, i)})$.

Then there exists $L \in [M]^\infty$ with $F \sqsubseteq L$ such that for every $L' \in [L]^\infty$ with $F \sqsubseteq L'$ and every $x^* \in B^*(L')$ there exists $y^* \in B^*(L')$ such that

$$|y^*(y_{F_{(\Delta, i_0)} \cup \{L'(|F|+1)\}})| \leq \frac{\delta_{|F_{(\Delta, i_0)}|+1}}{2^{|F|+1}}$$

PROOF. We set \mathcal{A} to be the set of all L' in $[M]^\infty$ with $F \sqsubset L'$ such that for every $x^* \in B^*(L')$ there exists $y^* \in B^*(L')$ such that

$$|y^*(y_{F_{(\Delta, i_0)} \cup \{L'(|F|+1)\}})| \leq \frac{\delta_{|F_{(\Delta, i_0)}|+1}}{2^{|F|+1}}$$

By property $(\Delta 2)$ it is easy to see that the set \mathcal{A} is open (actually it is clopen). Hence by Galvin-Prikry's theorem there exists L in $[M]^\infty$ with $F \sqsubset L$ such that either $L' \in \mathcal{A}$ for all $L' \in [L]^\infty$ with $F \sqsubset L'$, or $L' \notin \mathcal{A}$ for all $L' \in [L]^\infty$ with $F \sqsubset L'$. We will show that the second alternative is impossible.

Indeed suppose that $L' \notin \mathcal{A}$ for all $L' \in [L]^\infty$ with $F \sqsubset L'$. Then for each $L' \in [L]^\infty$ with $F \sqsubset L'$ we have that $B^*(L') \neq \emptyset$ and for every $y^* \in B^*(L')$

$$|y^*(y_{F_{(\Delta, i_0)} \cup \{L'(|F|+1)\}})| > \frac{\delta_{|F_{(\Delta, i_0)}|+1}}{2^{|F|+1}}$$

Let $P = L \setminus F = \{p_1 < p_2 < \dots\}$. For every $n \in \mathbb{N}$ and $1 \leq j \leq n$ we set

$$L'_{n,j} = F \cup \{p_j\} \cup \{p_l : l > n\}$$

By property $(\Delta 3)$ we have that for every $n \in \mathbb{N}$ and $1 \leq j_1, j_2 \leq n$, $B^*(L'_{n,j_1}) = B^*(L'_{n,j_2})$. Hence for every $n \in \mathbb{N}$ there exists $y_n^* \in B_{X^*}$ such that

$$|y_n^*(y_{F_{(\Delta, i_0)} \cup \{p_j\}})| = |y_n^*(y_{F_{(\Delta, i_0)} \cup \{L'_{n,j}(|F|+1)\}})| > \frac{\delta_{|F_{(\Delta, i_0)}|+1}}{2^{|F|+1}}$$

for all $1 \leq j \leq n$. Let y^* be a w^* -limit of $(y_n^*)_{n \in \mathbb{N}}$. Then

$$|y^*(y_{F_{(\Delta, i_0)} \cup \{p_j\}})| \geq \frac{\delta_{|F_{(\Delta, i_0)}|+1}}{2^{|F|+1}}$$

which is a contradiction since $(y_{F_{(\Delta, i_0)} \cup \{n\}})_{n > F_{(\Delta, i_0)}}$ is weakly null. \square

Corollary 7.12. Let $F \in [\mathbb{N}]^{<\infty}$ be \mathfrak{G} -free and $M \in [N]^\infty$ with $F \sqsubseteq M$. Then there exists $L \in [M]^\infty$ with $F \sqsubset L$ such that for every $L' \in [M]^\infty$ with $F \sqsubset L'$ and $x^* \in B_{X^*}$ there exists $y^* \in B_{X^*}$ satisfying the following.

- (i) For all $1 \leq i \leq k$ and $t \sqsubseteq F_{(\Delta, i)}$, $|x^*(y_t) - y^*(y_t)| \leq \frac{\delta_{|t|}}{2^{|F|+2}}$.
- (ii) For all $1 \leq i \leq k$ with $i \neq i_0$,

$$\left| (x^* - y^*) \left(\sum_{F_{(\Delta, i)} \sqsubset t \sqsubseteq v'_i} y_t \right) \right| \leq \frac{\varepsilon}{2^{|F|+2}}$$

where $v'_i = I_{\mathcal{G}_i}(L'_{(\Delta, i)})$.

- (iii) $|y^*(y_{F_{(\Delta, i_0)} \cup \{L'(|F|+1)\}})| \leq \frac{\delta_{|F_{(\Delta, i_0)}|+1}}{2^{|F|+1}}$.

PROOF. For every $1 \leq i \leq k$ and $t \sqsubseteq F_{(\Delta, i)}$ let A_t be a $\frac{\delta_{|t|}}{2^{|F|+3}}$ -net of $[-2K, 2K]$. For every $1 \leq i \leq k$ with $i \neq i_0$ let B_i be a $\frac{\varepsilon}{2^{|F|+3}}$ -net of $[-2K, 2K]$. Let

$$(((a_t^q)_{t \sqsubseteq F_{(\Delta, i)}})_{i=1}^k, (b_i^q)_{i=1, i \neq i_0}^m)_{q=1}^m$$

be an enumeration of the set $(\prod_{i=1}^k (\prod_{t \sqsubseteq F_{(\Delta, i)}} A_t)) \times (\prod_{i=1, i \neq i_0}^m B_i)$. We set $L_0 = M$ and inductively for $q = 1, \dots, m$, using Sublemma 7.11 for $((a_t^q)_{t \sqsubseteq F_{(\Delta, i)}})_{i=1}^k$ and $(b_i^q)_{i=1, i \neq i_0}^m$, we construct a decreasing sequence of infinite subsets $(L_q)_{q=1}^m$ of L_0 with $F \sqsubset L_q$, for all $1 \leq q \leq m$. It is easy to check that L_m is the desired set. \square

PROOF OF LEMMA 7.8. We may suppose that $(\delta_n)_{n=1}^\infty$ is a decreasing sequence and $\sum_{n=1}^\infty \delta_n < \varepsilon$. Let also $\delta_1 < \delta_0 < \varepsilon/2$ and $M = M_1 \in [N]^\infty$. Let F_1 be the \sqsubseteq -minimal initial segment of M_1 which is \mathfrak{G} -free. Applying Corollary 7.12 we obtain $M_2 \in [M_1]^\infty$ with $F_1 \sqsubset M_2$ satisfying the conclusion of Corollary 7.12 for $F = F_1$ and $L = M_2$. In the same way we construct by induction a sequence $(F_n)_n \in \mathbb{N}$ in $[M_1]^{<\infty}$ and a sequence $(M_n)_{n \in \mathbb{N}}$ in $[M_1]^\infty$ satisfying the following for every $n \in \mathbb{N}$.

- (i) For every $n \in \mathbb{N}$, $F_n \sqsubset M_n$.
- (ii) For every $n \in \mathbb{N}$, $F_n \sqsubset F_{n+1}$ and $M_{n+1} \in [M_n]^\infty$.
- (iii) For every $n \in \mathbb{N}$, the conclusion of Corollary 7.12 is satisfied for $F = F_n$ and $L = M_{n+1}$.

We set $L = \cup_{n=1}^\infty F_n$. We claim that L is $(\varepsilon, (\delta_n)_{n \in \mathbb{N}})$ -good. Indeed, let $x^* \in B_{X^*}$ and set $y_0^* = x^*$. For every $1 \leq i \leq k$ let v_i be the unique element of \mathcal{G}_i such that $v_i \sqsubset L_{(\Delta, i)}$. For every $n \in \mathbb{N}$ we set

$$t_i^n = v_i \cap F_n = v_i \cap (F_n)_{(\Delta, i)}$$

and for every $1 \leq i \leq k$ and $v_i \sqsubset t \sqsubset L_{(\Delta, i)}$ we set

$$n_t = \min \{n \in \mathbb{N} : t \subseteq L(|F_n| + 1)\}$$

Notice that

$$t = (F_{n_t})_{(\Delta, i)} \cup \{L(|F_{n_t}| + 1)\}$$

for all $1 \leq i \leq k$ and $v_i \sqsubset t \sqsubset L_{(\Delta, i)}$. Finally let, for every $n \in \mathbb{N}$, $i_n = i_\Delta(|F_n| + 1)$.

By the construction of $(F_n)_{n \in \mathbb{N}}$ we may inductively choose a sequence $(y_n^*)_{n \in \mathbb{N}}$ in B_{X^*} such that for every $n \in \mathbb{N}$ the following are satisfied:

- (i) For all $1 \leq i \leq k$ and $t \sqsubseteq (F_n)_{(\Delta, i)}$, $|y_{n-1}^*(y_t) - y_n^*(y_t)| \leq \frac{\delta_{|t|}}{2^{|F_n|+2}}$.
- (ii) For all $1 \leq i \leq k$ with $i \neq i_n$,

$$\left| (y_{n-1}^* - y_n^*)(x_{v_i} - x_{t_i^n}) \right| = \left| (y_{n-1}^* - y_n^*) \left(\sum_{(F_n)_{(\Delta, i)} \sqsubset t \sqsubseteq v_i^n} y_t \right) \right| \leq \frac{\varepsilon}{2^{|F_n|+2}}$$

$$(iii) \quad |y_n^*(y_{(F_n)_{(\Delta, i_n)} \cup \{L(|F_n|+1)\}})| \leq \frac{\delta_{|(F_n)_{(\Delta, i_n)} \cup \{L(|F_n|+1)\}}|+1}{2^{|F_n|+1}}.$$

By (i) and (ii) we have that for every $n \in \mathbb{N}$ and $1 \leq i \leq k$,

$$\begin{aligned} |(y_{n-1}^* - y_n^*)(x_{v_i})| &\leq |(y_{n-1}^* - y_n^*)(x_{v_i} - x_{t_i^n})| + \sum_{t \sqsubseteq t_i^n} |(y_{n-1}^* - y_n^*)(y_t)| \\ &\leq \frac{\varepsilon}{2^{|F_n|+2}} + \sum_{t \sqsubseteq t_i^n} \frac{\delta_{|t|}}{2^{|F_n|+2}} < \frac{\varepsilon}{2^{|F_n|+1}} \end{aligned}$$

This yields that $|(x^* - y_n^*)(x_{v_i})| < \varepsilon$, for all $n \in \mathbb{N}$. By (iii) and the definition of the natural number n_t , we have that for every $1 \leq i \leq k$ and every $v_i \sqsubset t \sqsubset L_{(\Delta, i)}$,

$$|y_{n_t}^*(y_t)| \leq \frac{\delta_{|(F_{n_t})_{(\Delta, i_0)}|+1}}{2^{|F_{n_t}|+1}} = \frac{\delta_{|t|}}{2^{|F_{n_t}|+1}}$$

Hence by (i) and (iii) for every $1 \leq i \leq k$, every $v_i \sqsubset t \sqsubset L_{(\Delta, i)}$ and $n > n_t$, we have that $t \sqsubseteq (F_m)_{(\Delta, i)}$, for all $n_t < m \leq n$, and

$$\begin{aligned} |y_n^*(y_t)| &\leq |y_{n_t}^*(y_t)| + \sum_{m=n_t+1}^n |(y_{m-1}^* - y_m^*)(y_t)| \\ &\leq \frac{\delta_{|t|}}{2^{|F_{n_t}|+1}} + \frac{\delta_{|t|}}{2^{|F_m|+2}} \leq \delta_{|t|} \end{aligned}$$

Let y^* be a w^* -limit of $(y_n)_{n \in \mathbb{N}}$. Then $y^* \in B_{X^*}$ and for each $1 \leq i \leq k$,

- (a) $|(x^* - y^*)(x_{v_i})| \leq \varepsilon$.
- (b) For all $t \in [L]^{<\infty}$ with $v_i \sqsubset t \sqsubset L_{(\Delta, i)}$, for some $i \in \{1, \dots, k\}$, we have that $|y^*(y_t)| \leq \delta_{|t|}$.

Since for every $x^* \in B_{X^*}$ there exists $y^* \in B_{X^*}$ satisfying (a) and (b), we have that L is $(\varepsilon, (\delta_n)_{n \in \mathbb{N}})$ -good. \square

2. Dominated spreading models

Theorem 7.13. Let \mathcal{F}, \mathcal{G} be regular thin families and $N \in [\mathbb{N}]^\infty$ such that $\mathcal{G} \upharpoonright N \sqsubset \mathcal{F} \upharpoonright N$. Let $(x_s)_{s \in \mathcal{F}}$ be a bounded \mathcal{F} -sequence in a Banach space X such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright N}$ is subordinated (with respect to the weak topology) and let $\widehat{\varphi} : \widehat{F} \upharpoonright N \rightarrow (X, w)$ be the continuous map witnessing this. Let for every $u \in \mathcal{G} \upharpoonright N$, $z_v = \widehat{\varphi}(v)$. Suppose that $(x_s)_{s \in \mathcal{F} \upharpoonright N}$ and $(z_v)_{v \in \mathcal{G} \upharpoonright N}$ generate $(e_n^1)_{n \in \mathbb{N}}$ as an \mathcal{F} -spreading model and $(e_n^2)_{n \in \mathbb{N}}$ as a \mathcal{G} -spreading model respectively. Then for every $k \in \mathbb{N}$ and $a_1, \dots, a_k \in \mathbb{R}$ we have that

$$\left\| \sum_{j=1}^k a_j e_j^2 \right\| \leq \left\| \sum_{j=1}^k a_j e_j^1 \right\|$$

The proof of the above theorem relies on a series of lemmas which are presented below.

Lemma 7.14. Let $k \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists $M \in [N]^\infty$ such that for every plegma k -tuple $(s_j)_{j=1}^k$ in $\mathcal{F} \upharpoonright M$ and $a_1, \dots, a_k \in \mathbb{R}$, we have that

$$\left\| \sum_{j=1}^k a_j x_{s_j} \right\| \geq \left\| \sum_{j=1}^k a_j z_{v_j} \right\| - \varepsilon \sum_{j=1}^k |a_j|$$

where for each $1 \leq j \leq k$, v_j is the unique element in \mathcal{G} such that $v_j \sqsubset s_j$.

PROOF. We define a bounded weakly null tree $(y_t)_{t \in [\mathbb{N}]^{<\infty}}$ as follows. For each $t \notin \widehat{\mathcal{F}} \upharpoonright N$ we set $y_t = 0$. We also set $y_\emptyset = \widehat{\varphi}(\emptyset)$ and for each nonempty $t \in \widehat{\mathcal{F}} \upharpoonright N$ we set $y_t = \widehat{\varphi}(t) - \widehat{\varphi}(t^*)$, where $t^* = t \setminus \{\max t\}$. It is immediate that $(y_t)_{t \in [\mathbb{N}]^{<\infty}}$ is a bounded weakly null tree. Let $\varepsilon > 0$ and $(\delta_n)_{n \in \mathbb{N}}$ be a decreasing null sequence of positive reals such that $\sum_{n=1}^\infty \delta_n < \frac{\varepsilon}{2}$. Let $k \in \mathbb{N}$ and $\Delta^k = (N_k^{(1)}, \dots, N_k^{(k)})$ be the partition of \mathbb{N} into disjoint infinite sets such that $N_k^{(j)} = \{n \in \mathbb{N} : n = j(\bmod k)\}$, for all $1 \leq j < k$, and $N_k^{(k)} = \{n \in \mathbb{N} : n = 0(\bmod k)\}$. Applying Theorem 7.6

for $k, \Delta^k, N, (\mathcal{G}, \dots, \mathcal{G}), (y_t)_{t \in [\mathbb{N}]^{<\infty}}, \frac{\varepsilon}{2}$ and $(\delta_n)_{n \in \mathbb{N}}$, we obtain $M' \in [N]^\infty$ such that every $L \in [M']^\infty$ is $(\frac{\varepsilon}{2}, (\delta_n)_{n \in \mathbb{N}})$ -good.

We set $M = \{M'(k \cdot n) : n \in \mathbb{N}\}$. It is easy to check that for every plegma k -tuple $(s_j)_{j=1}^k$ in $\mathcal{F} \upharpoonright M$ there exists $L \in [M']^\infty$ such that $s_j \sqsubset L_{(\Delta^k, j)}$, for all $1 \leq j \leq k$. The set M is the desired one.

Indeed, let $a_1, \dots, a_k \in \mathbb{R}$ and $(s_j)_{j=1}^k$ be a plegma k -tuple in $\mathcal{F} \upharpoonright M$. Let $L \in [M']^\infty$ such that $s_j \sqsubset L_{(\Delta^k, j)}$, for all $1 \leq j \leq k$. Let also for each $1 \leq j \leq k$, v_j the unique element in \mathcal{G} such that $v_j \sqsubset s_j$. Then for every $x^* \in B_{X^*}$ there exists $y^* \in B_{X^*}$ such that for every $1 \leq j \leq k$

- (i) $|x^*(z_{v_j}) - y^*(z_{v_j})| \leq \frac{\varepsilon}{2}$ and
- (ii) $|y^*(y_t)| \leq \delta_{|t|}$, for all $t \sqsubset v_j$.

By (ii) and the choice of the family $(y_t)_{t \in [\mathbb{N}]^{<\infty}}$ it is easy to see that

$$|y^*(x_{s_j} - z_{v_j})| \leq \frac{\varepsilon}{2}$$

for all $1 \leq j \leq k$. Therefore we have that

$$\begin{aligned} (11) \quad \left| y^* \left(\sum_{j=1}^k a_j x_{s_j} \right) \right| &\geq \left| y^* \left(\sum_{j=1}^k a_j z_{v_j} \right) \right| - \frac{\varepsilon}{2} \sum_{j=1}^k |a_j| \\ &\geq \left| x^* \left(\sum_{j=1}^k a_j z_{v_j} \right) \right| - \varepsilon \sum_{j=1}^k |a_j| \end{aligned}$$

Since for every $x^* \in B_{X^*}$ there exists $y^* \in B_{X^*}$ which satisfies (11), we have that

$$\left\| \sum_{j=1}^k a_j x_{s_j} \right\| \geq \left\| \sum_{j=1}^k a_j z_{v_j} \right\| - \varepsilon \sum_{j=1}^k |a_j|$$

□

Lemma 7.15. For every $(\varepsilon_n)_{n \in \mathbb{N}}$ decreasing null sequence of positive reals, there exists $M \in [N]^\infty$ such that for every $k \leq l$ in \mathbb{N} , every plegma k -tuple $(s_j)_{j=1}^k$ in $\mathcal{F} \upharpoonright M$ with $s_1(1) = M(l)$ and $a_1, \dots, a_k \in \mathbb{R}$ we have that

$$\left\| \sum_{j=1}^k a_j x_{s_j} \right\| \geq \left\| \sum_{j=1}^k a_j z_{v_j} \right\| - \varepsilon_l \sum_{j=1}^k |a_j|$$

where for each $1 \leq j \leq k$, v_j is the unique element in \mathcal{G} such that $v_j \sqsubset s_j$.

PROOF. Using Lemma 7.14 we inductively construct a decreasing sequence $(M_l)_{l \in \mathbb{N}}$ in $[N]^\infty$ such that for every $l \in \mathbb{N}$ we have that for every $1 \leq k \leq l$, every plegma k -tuple $(s_j)_{j=1}^k$ in $\mathcal{F} \upharpoonright M_l$ and $a_1, \dots, a_k \in \mathbb{R}$

$$\left\| \sum_{j=1}^k a_j x_{s_j} \right\| \geq \left\| \sum_{j=1}^k a_j z_{v_j} \right\| - \varepsilon_l \sum_{j=1}^k |a_j|$$

where for each $1 \leq j \leq k$, v_j is the unique element in \mathcal{G} such that $v_j \sqsubset s_j$. Let $M \in [N]^\infty$ such that $M(l) \in M_l$, for all $l \in \mathbb{N}$. It is easy to check that M is the desired set. □

By the above lemma the proof of Theorem 7.13 is immediate.

3. Plegma block generated c_0 spreading models

Theorem 7.16. Let X be a Banach space with a Schauder basis such that $\mathcal{SM}^{wrc}(X)$ contains a sequence equivalent to the usual basis of c_0 . Let ξ_0 be the minimum countable ordinal ξ such that $\mathcal{SM}_\xi^{wrc}(X)$ contains a sequence equivalent to the usual basis of c_0 .

Let \mathcal{F} be a regular thin family of order ξ_0 , $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ a weakly relatively compact \mathcal{F} sequence in X such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates c_0 as an \mathcal{F} -spreading model. Then for every sequence $(\delta_n)_{n \in \mathbb{N}}$ of positive real numbers there exist $N \in [M]^\infty$ and an \mathcal{F} -subsequence $(z_s)_{s \in \mathcal{F} \upharpoonright N}$ such that the following are satisfied:

- (i) For every $s \in \mathcal{F} \upharpoonright N$ if $\min s = N(l)$, then $\|z_s - x_s\| < \delta_l$.
- (ii) For every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright N$ we have that $z_{s_1} < z_{s_2}$, that is $(z_s)_{s \in \mathcal{F} \upharpoonright N}$ is a plegma block subsequence.

PROOF. We may suppose that \mathcal{F} is very large in M . By Proposition 4.14 there exists $L \in [M]^\infty$ such that the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ admits a generic assignment $(\hat{\varphi}, (\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L}, (\tilde{y}_t)_{t \in \hat{\mathcal{F}} \upharpoonright L})$. Since $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ generates an \mathcal{F} -spreading model which is Schauder basic and not equivalent to the usual basis of ℓ^1 , by Lemma 4.7, we have that $\hat{\varphi}(\emptyset) = \tilde{y}_\emptyset = \tilde{x}_\emptyset = 0$.

Claim: For every $\delta > 0$ and $L' \in [L]^\infty$ there exists $L'' \in [L']^\infty$ such that for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L''$ we have that

$$\left\| \sum_{t \sqsubseteq s_2/s_1} \tilde{y}_t \right\| < \delta$$

where $s_2/s_1 = s_2 \cap \{1, \dots, \max s_1\}$.

PROOF OF THE CLAIM. Let $\delta > 0$ and $L' \in [L]^\infty$. Then by Proposition 1.12 there exists $L'' \in [L']^\infty$ such that either

$$\left\| \sum_{t \sqsubseteq s_2/s_1} \tilde{y}_t \right\| < \delta$$

for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L''$, or

$$\left\| \sum_{t \sqsubseteq s_2/s_1} \tilde{y}_t \right\| \geq \delta$$

for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L''$. We will show that the second alternative is impossible.

Indeed, suppose that the second case holds. We set $\mathcal{G} = \mathcal{F}/L''$ (see Definition 6.6). By Lemma 6.7 we have that $o(\mathcal{G}) < o(\mathcal{F}) = \xi_0$. We consider the first component $((x_s^{(1, \mathcal{G})})_{s \in \mathcal{F} \upharpoonright L''}, \hat{\varphi}^{(1, \mathcal{G})}, (\tilde{x}_s^{(1, \mathcal{G})})_{s \in \mathcal{F} \upharpoonright L''}, (\tilde{y}_t^{(1, \mathcal{G})})_{t \in \hat{\mathcal{F}} \upharpoonright L''})$ of the \mathcal{G} -splitting of $((x_s)_{s \in \mathcal{F} \upharpoonright L''}, \hat{\varphi}, (\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright L''}, (\tilde{y}_t)_{t \in \hat{\mathcal{F}} \upharpoonright L''})$ (see Definition 6.3). It is easy to see that the \mathcal{G} -subsequence $(x_s^{(1, \mathcal{G})})_{s \in \mathcal{F} \upharpoonright L''}$ is seminormalized and weakly relatively compact. We pass to $L_1 \in [L'']^\infty$ such that the \mathcal{G} -subsequence $(x_s^{(1, \mathcal{G})})_{s \in \mathcal{F} \upharpoonright L_1}$ generates a \mathcal{G} -spreading model $(e_n)_{n \in \mathbb{N}}$. Since $(x_s^{(1, \mathcal{G})})_{s \in \mathcal{F} \upharpoonright L_1}$ is seminormalized and $\hat{\varphi} = \emptyset$, by Theorem 3.32 we have that $(e_n)_{n \in \mathbb{N}}$ is a non trivial unconditional sequence. The latter, by Proposition 7.13, yields that $(e_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of c_0 . By Lemma 6.5 we contradict the assumption of the minimality of ξ_0 . \square

Using the above claim we inductively construct a decreasing sequence $(L_n)_{n \in \mathbb{N}}$ in $[L]^\infty$ such that for every $n \in \mathbb{N}$ and every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L_n$ we have that

$$\left\| \sum_{t \sqsubseteq s_2/s_1} \tilde{y}_t \right\| < \delta_n$$

Let $L_\infty \in [L]^\infty$ such that $L_\infty(n) \in L_n$, for all $n \in \mathbb{N}$. We set $N = L_\infty(2\mathbb{N})$. Let $(z_s)_{s \in \mathcal{F} \upharpoonright N}$ defined as in Lemma 6.12 (with N in place of M). Then by the above we have that $(z_s)_{s \in \mathcal{F} \upharpoonright N}$ is the desired \mathcal{F} -subsequence. \square

Remark 7.17. It is easy to see that $(z_s)_{s \in \mathcal{F} \upharpoonright N}$ generates the same \mathcal{F} -spreading model as the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$.

Corollary 7.18. Let X be a Banach space with a Schauder basis. For every $\xi < \omega_1$ we have that if $\mathcal{SM}_\xi^{wrc}(X)$ contains a sequence equivalent to the usual basis of c_0 , then X admits c_0 as a plegma block generated ξ -order spreading model.

4. Duality of c_0 and ℓ^1 spreading models

Definition 7.19. Let X be a Banach space, $c > 0$, \mathcal{F} a regular thin family, $M \in [\mathbb{N}]^\infty$, $(x_s)_{s \in \mathcal{F}}$ a normalized \mathcal{F} -sequence in X and $(x_s^*)_{s \in \mathcal{F}}$ a bounded \mathcal{F} -sequence in X^* . We will say that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ and $(x_s^*)_{s \in \mathcal{F} \upharpoonright M}$ are ℓ^1 -associated over c if for every $k \in \mathbb{N}$, $a_1, \dots, a_k \in \mathbb{R}$ and every plegma k -tuple $(s_j)_{j=1}^k$ in $\mathcal{F} \upharpoonright M$ with $s_1(1) \geq M(k)$ we have that

$$\sum_{j=1}^k a_j x_{s_j}^* \left(\sum_{j=1}^k \text{sign}(a_j) x_{s_j} \right) \geq c \sum_{j=1}^k |a_j|$$

We will say that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ and $(x_s^*)_{s \in \mathcal{F} \upharpoonright M}$ are ℓ^1 -associated if there exists $c > 0$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ and $(x_s^*)_{s \in \mathcal{F} \upharpoonright M}$ are ℓ^1 -associated over c .

Remark 7.20. Suppose that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ and $(x_s^*)_{s \in \mathcal{F} \upharpoonright M}$ are ℓ^1 -associated. Then if $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates c_0 as an \mathcal{F} -spreading model, then it is easy to see that every \mathcal{F} -spreading model admitted by $(x_s^*)_{s \in \mathcal{F} \upharpoonright M}$ is ℓ^1 .

Definition 7.21. Let X be a Banach space, \mathcal{F} a regular thin family, $M \in [\mathbb{N}]^\infty$, $(x_s)_{s \in \mathcal{F}}$ a normalized \mathcal{F} -sequence in X and $(x_s^*)_{s \in \mathcal{F}}$ a bounded \mathcal{F} -sequence in X^* . The \mathcal{F} -subsequence $(x_s^*)_{s \in \mathcal{F} \upharpoonright M}$ is called biorthogonal to $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ if for every $k \in \mathbb{N}$ and every plegma k -tuple $(s_j)_{j=1}^k$ in $\mathcal{F} \upharpoonright M$ with $s_1(1) \geq M(k)$ we have that

$$x_{s_j}^*(x_{s_i}) = \delta_{ij}, \text{ for all } i, j \in \{1, \dots, k\}$$

Remark 7.22. It is immediate that if $(x_s^*)_{s \in \mathcal{F} \upharpoonright M}$ is biorthogonal to $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ then $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ and $(x_s^*)_{s \in \mathcal{F} \upharpoonright M}$ are ℓ^1 -associated over 1.

Lemma 7.23. Let X be a Banach space with a Schauder basis $(e_n)_{n \in \mathbb{N}}$, \mathcal{F} a regular thin family, $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ a normalized plegma block \mathcal{F} -sequence in X . Then there exists an \mathcal{F} -subsequence $(x_s^*)_{s \in \mathcal{F} \upharpoonright M}$ biorthogonal to $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ with respect to $(e_n^*)_{n \in \mathbb{N}}$.

PROOF. By Hahn-Banach theorem, for every $s \in \mathcal{F} \upharpoonright M$ there exists $\tilde{x}_s^* \in B_{X^*}$ such that $\tilde{x}_s^*(x_s) = \|x_s\| = 1$. For every $s \in \mathcal{F} \upharpoonright M$ we set

$$x_s^* = \sum_{n \in \text{range}(x_s)} \tilde{x}_s^*(e_n) e_n^*$$

where $\text{range}(x_s) = \{\min(\text{supp}(x_s)), \dots, \max(\text{supp}(x_s))\}$. It is easy to check that $(x_s^*)_{s \in \mathcal{F} \upharpoonright M}$ satisfies the conclusion of the lemma. \square

Proposition 7.24. Let X be a Banach space with a Schauder basis $(e_n)_{n \in \mathbb{N}}$. If for some $\xi < \omega_1$ the set $\mathcal{SM}_\xi^{wrc}(X)$ contains a sequence equivalent to the usual basis of c_0 , then $\overline{\langle (e_n^*)_{n \in \mathbb{N}} \rangle}$ admits ℓ^1 as a plegma block generated ξ -order spreading model.

PROOF. By Corollary 7.18 there exist a regular thin family \mathcal{F} of order ξ , $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in X such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ plegma block generates c_0 as an \mathcal{F} -spreading model. We may also assume that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ is normalized. By Lemma 7.23 there exists a bounded \mathcal{F} -subsequence $(x_s^*)_{s \in \mathcal{F} \upharpoonright M}$ biorthogonal to $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ which is plegma block with respect to $(e_n^*)_{n \in \mathbb{N}}$. By Remarks 7.22 and 7.20 we have that $(x_s^*)_{s \in \mathcal{F} \upharpoonright M}$ admits ℓ^1 as an \mathcal{F} -spreading model. \square

Lemma 7.25. Let X be a Banach space, \mathcal{F} a regular thin family, $M \in [\mathbb{N}]^\infty$, $(x_s)_{s \in \mathcal{F}}$ a normalized \mathcal{F} -sequence in X and $(x_s^*)_{s \in \mathcal{F}}$ a bounded \mathcal{F} -sequence in X^* such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ and $(x_s^*)_{s \in \mathcal{F} \upharpoonright M}$ are ℓ^1 -associated over c , for some $c > 0$. Let $(\delta_n)_{n \in \mathbb{N}}$ a sequence of positive reals and $(z_s)_{s \in \mathcal{F}}$ a normalized \mathcal{F} -sequence in X such that $\|x_s - z_s\| \leq \delta_n$, for all $n \in \mathbb{N}$ and $s \in \mathcal{F} \upharpoonright M$ with $\min s = M(n)$. If $\sum_{n=1}^\infty \delta_n < \frac{c}{2K}$, where $K = \sup\{\|x_s^*\| : s \in \mathcal{F} \upharpoonright M\}$ then $(z_s)_{s \in \mathcal{F} \upharpoonright M}$ and $(x_s^*)_{s \in \mathcal{F} \upharpoonright M}$ are ℓ^1 -associated over $c/2$.

PROOF. Indeed, for every $k \in \mathbb{N}$, $a_1, \dots, a_k \in \mathbb{R}$ and every plegma k -tuple $(s_j)_{j=1}^k$ in $\mathcal{F} \upharpoonright M$ with $s_1(1) \geq M(k)$ we have that

$$\begin{aligned} & \sum_{j=1}^k a_j x_{s_j}^* \left(\sum_{j=1}^k \text{sign}(a_j) z_{s_j} \right) \\ & \geq \sum_{j=1}^k a_j x_{s_j}^* \left(\sum_{j=1}^k \text{sign}(a_j) x_{s_j} \right) - \sum_{j=1}^k |a_j| \cdot \|x_{s_j}^*\| \left\| \sum_{i=1}^k \|z_{s_i} - x_{s_i}\| \right\| \\ & \geq \frac{c}{2} \sum_{j=1}^k |a_j| \end{aligned}$$

\square

Theorem 7.26. Let X be a Banach space. If for some $\xi < \omega_1$ the set $\mathcal{SM}_\xi^{wrc}(X)$ contains a sequence equivalent to the usual basis of c_0 , then X^* admits ℓ^1 as a ξ -order spreading model.

PROOF. Let ξ_0 be the minimum countable ordinal ξ such that $\mathcal{SM}_\xi^{wrc}(X)$ contains a sequence equivalent to the usual basis of c_0 . Notice that it suffices to prove the theorem for $\xi = \xi_0$. Let \mathcal{F} be a regular thin family of order ξ_0 , $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ a weakly relatively compact \mathcal{F} -sequence in X such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates c_0 as an \mathcal{F} -spreading model. Observe that we may also assume that $(x_s)_{s \in \mathcal{F}}$ is normalized. Let Y be a separable (closed) subspace of X such that $\{x_s : s \in \mathcal{F}\} \subseteq Y$ and $T : Y \rightarrow C[0, 1]$ an isometric (linear) embedding. Let $(e_n)_{n \in \mathbb{N}}$ be a Schauder basis of $C[0, 1]$. Then $(T(x_s))_{s \in \mathcal{F}}$ is a normalized weakly relatively compact \mathcal{F} -sequence in $C[0, 1]$. Let $(\delta_n)_{n \in \mathbb{N}}$ be a sequence of positive reals such that $\sum_{n=1}^\infty \delta_n < \infty$. By Theorem 7.16 there exist $L_1 \in [M]^\infty$ and $(\tilde{z}_s)_{s \in \mathcal{F} \upharpoonright L_1}$ such that the following are satisfied:

- (i) For every $s \in \mathcal{F} \upharpoonright L_1$, if $\min s = L_1(l)$, then $\|\tilde{z}_s - T(x_s)\| < \frac{\delta_l}{2}$.
- (ii) For every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L_1$ we have that $\tilde{z}_{s_1} < \tilde{z}_{s_2}$.

For every $s \in \mathcal{F} \upharpoonright L_1$ we set $z_s = \frac{\tilde{z}_s}{\|\tilde{z}_s\|}$. Then $(z_s)_{s \in \mathcal{F} \upharpoonright L_1}$ is a normalized \mathcal{F} -subsequence such that

- (a) For every $s \in \mathcal{F} \upharpoonright L_1$, if $\min s = L_1(l)$, then $\|z_s - T(x_s)\| < \delta_l$.
- (b) For every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L_1$ we have that $z_{s_1} < z_{s_2}$.

By Lemma 7.23 there exists an \mathcal{F} -subsequence $(z_s^*)_{s \in \mathcal{F} \upharpoonright L_1}$ in $(C[0, 1])^*$ biorthogonal to $(z_s)_{s \in \mathcal{F} \upharpoonright L_1}$. By Remark 7.22 we have that $(z_s)_{s \in \mathcal{F} \upharpoonright L_1}$ and $(z_s^*)_{s \in \mathcal{F} \upharpoonright L_1}$ are ℓ^1 -associated over 1. Let $K = \sup\{\|z_s^*\| : s \in \mathcal{F} \upharpoonright L_1\}$, $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty} \delta_n < \frac{1}{2K}$ and $L = \{L_1(n) : n \geq n_0\}$. By Lemma 7.25 $(T(x_s))_{s \in \mathcal{F} \upharpoonright L}$ and $(z_s^*)_{s \in \mathcal{F} \upharpoonright L}$ are ℓ^1 -associated over $\frac{1}{2}$. This easily yields that $(x_s)_{s \in \mathcal{F} \upharpoonright L}$ and $(T^*(z_s^*))_{s \in \mathcal{F} \upharpoonright L}$ are ℓ^1 -associated over $\frac{1}{2}$. Notice that $(T^*(z_s^*))_{s \in \mathcal{F} \upharpoonright L}$ is a bounded \mathcal{F} -subsequence in Y^* . For every $s \in \mathcal{F} \upharpoonright L$, by Hahn-Banach theorem there exists $x_s^* \in X^*$ such that $\|x_s^*\| = \|T^*(z_s^*)\|$ and $x_s^*|_Y = T^*(z_s^*)|_Y$. Therefore $(x_s^*)_{s \in \mathcal{F} \upharpoonright L}$ is a bounded \mathcal{F} -subsequence in X^* and $(x_s)_{s \in \mathcal{F} \upharpoonright L}$, $(x_s^*)_{s \in \mathcal{F} \upharpoonright L}$ are ℓ^1 -associated (over $\frac{1}{2}$). Hence by Remark 7.20 we have that $(x_s^*)_{s \in \mathcal{F} \upharpoonright L}$ admits ℓ^1 as an \mathcal{F} -spreading model. \square

CHAPTER 8

Establishing the hierarchy of spreading models

In this chapter we deal with the problem of the existence of spaces X admitting ℓ^1 as ξ -order spreading model but not less. We present two examples. The first one answers the problem for $\xi < \omega$ and the second concerns transfinite countable ordinals.

1. Spaces admitting ℓ^1 as $k+1$ but not as k order spreading model

In this section we will present for each $k \in \mathbb{N}$ a Banach space \mathfrak{X}_{k+1} , having an unconditional basis $(e_s)_{s \in [\mathbb{N}]^{k+1}}$ which generates an ℓ^1 spreading model of $(k+1)$ -order and is not $(k+1)$ -Cesàro summable to any x_0 in \mathfrak{X}_{k+1} . Moreover the space \mathfrak{X}_{k+1} does not contain any k -order ℓ^1 spreading model. In particular the space \mathfrak{X}_{k+1} shows that the non $(k+1)$ -Cesàro summability of a $[\mathbb{N}]^{k+1}$ -sequence does not yield any further information concerning ℓ^1 spreading models beyond the conclusion of Theorem 6.28.

1.1. The definition of the space \mathfrak{X}_{k+1} . The space \mathfrak{X}_{k+1} will be defined as the completion of $c_{00}([\mathbb{N}]^{k+1})$ under a certain norm. To this end we need the following definition.

Definition 8.1. Let $k \in \mathbb{N}$. A family $E \subseteq [\mathbb{N}]^{k+1}$ is said to be *allowable* if there exist $F_1 < \dots < F_{k+1}$ subsets of \mathbb{N} with $|F_1| = \dots = |F_{k+1}|$ and $|F_1| \leq \min F_1$ such that $E \subseteq F_1 \times \dots \times F_{k+1}$.

For instance, for every plegma l -tuple $(s_j)_{j=1}^l$ in $[\mathbb{N}]^{k+1}$ the set $E = \{s_1, \dots, s_l\}$ is allowable. For $x = \sum_{s \in [\mathbb{N}]^{k+1}} x(s)e_s$ in $c_{00}([\mathbb{N}]^{k+1})$ we set

$$\|x\| = \sup \left\{ \left(\sum_{j=1}^n \|x|_{E_i}\|_1^2 \right)^{\frac{1}{2}} : n \in \mathbb{N}, (E_i)_{i=1}^n \text{ are disjoint} \right. \\ \left. \text{and each } E_i \text{ is allowable} \right\}$$

where $\|x|_{E_i}\|_1 = \sum_{s \in E_i} |x(s)|$.

The space \mathfrak{X}_{k+1} is defined to be the completion of $(c_{00}([\mathbb{N}]^{k+1}), \|\cdot\|)$. It is easy to check that a norming set for the space \mathfrak{X}_{k+1} is the set W_{k+1} which is defined as follows. We first set

$$W_{k+1}^0 = \left\{ \sum_{s \in E} \pm e_s^* : E \text{ is allowable} \right\}$$

and then we define

$$W_{k+1} = \left\{ \sum_{i=1}^n \lambda_i f_i : n \in \mathbb{N}, \{f_i\}_{i=1}^n \subseteq W_{k+1}^0 \right. \\ \left. \text{with pairwise disjoint supports and } \sum_{i=1}^n \lambda_i^2 \leq 1 \right\}$$

It is immediate that the Hamel basis $(e_s)_{s \in [\mathbb{N}]^{k+1}}$ of $c_{00}([\mathbb{N}]^{k+1})$ consists an unconditional basis for the space \mathfrak{X}_{k+1} and that it generates isometrically a $(k+1)$ -order ℓ^1 spreading model.

As we have already mentioned we will show the following.

Theorem 8.2. The space \mathfrak{X}_{k+1} does not admit ℓ^1 as a spreading model of order l , for all $1 \leq l \leq k$.

The proof of Theorem 8.2 requires the next proposition.

Proposition 8.3. The space \mathfrak{X}_{k+1} does not admit any plegma disjointly generated ℓ^1 spreading model of order k .

The proof of Proposition 8.3 is postponed in the next subsection. Granting of this proposition we obtain the reflexivity of the space \mathfrak{X}_{k+1} and the proof of Theorem 8.2 as follows.

Corollary 8.4. The space \mathfrak{X}_{k+1} is reflexive.

PROOF. It is easy to check that the basis $(e_s)_{s \in [\mathbb{N}]^{k+1}}$ is boundedly complete which yields that c_0 is not contained in the space \mathfrak{X}_{k+1} . Moreover if ℓ^1 is embedded into \mathfrak{X}_{k+1} , then there exists a disjointly supported sequence $(x_n)_{n \in \mathbb{N}}$ equivalent to the usual basis of ℓ^1 , which contradicts Proposition 8.3. Hence, by James' theorem (c.f. [16]), the space \mathfrak{X}_{k+1} is reflexive. \square

PROOF OF THEOREM 8.2. By Corollary 2.16, it is enough to show that \mathfrak{X}_{k+1} does not admit any ℓ^1 spreading model of order k . Indeed, assume on the contrary. Let $(x_s)_{s \in [\mathbb{N}]^k}$ be a sequence in \mathfrak{X}_{k+1} which generates an ℓ^1 spreading model $(e_n)_{n \in \mathbb{N}}$ of order k . By Corollary 8.4, the sequence $(x_s)_{s \in [\mathbb{N}]^k}$ is weakly relatively compact. Hence by Theorem 4.15 there exist $M \in [\mathbb{N}]^\infty$ and a subsequence $(x'_s)_{s \in [M]^k}$ in \mathfrak{X}_{k+1} which generates an ℓ^1 spreading model and admits a disjointly generic decomposition. This contradicts Proposition 8.3. \square

We close the section by showing the following.

Proposition 8.5. Every subsequence of the basis $(e_s)_{s \in [\mathbb{N}]^{k+1}}$ of \mathfrak{X}_{k+1} is not $(k+1)$ -Cesàro summable to any x_0 in \mathfrak{X}_{k+1} .

PROOF. Since \mathfrak{X}_{k+1} is reflexive and $(e_s)_{s \in [\mathbb{N}]^{k+1}}$ is a Schauder basis of \mathfrak{X}_{k+1} , by part (iii) of Remark 3.2 we get that $(e_s)_{s \in [\mathbb{N}]^{k+1}}$ is weakly convergent to 0. Therefore by Remark 6.20 if for some $M \in [\mathbb{N}]^\infty$ the subsequence $(e_s)_{s \in [M]^{k+1}}$ was $(k+1)$ -Cesàro summable to some x_0 in \mathfrak{X}_{k+1} , then $x_0 = 0$. We will show that this is impossible. Indeed, let $M \in [\mathbb{N}]^\infty$. For every $n \in \mathbb{N}$ we set

$$y_n = \binom{(k+2)n}{k+1}^{-1} \sum_{s \in [M|(k+2)n]^{k+1}} e_s$$

and for every $1 \leq i \leq k+2$ we also set

$$F_i^n = \{M((i-1)n+1), \dots, M(in)\}$$

Moreover let

$$E_n = F_2^n \times \dots \times F_{k+2}^n \text{ and } \varphi_n = \sum_{s \in E_n} e_s^*$$

Then $\varphi_n \in W_{k+1}^0$ and $\|y_n\| \geq \varphi_n(y_n) = n^{k+1} \cdot \binom{(k+2)n}{k+1}^{-1} \xrightarrow{n \rightarrow \infty} \frac{(k+1)!}{(k+2)^{k+1}}$. \square

1.2. Proof of Proposition 8.3. We will need the following terminology.

- (a) Let $s_1, s_2 \in [\mathbb{N}]^{k+1}$. We say that s_1, s_2 are *allowable* if the set $\{s_1, s_2\}$ is allowable.
- (b) Let G_1, G_2 be subsets of $[\mathbb{N}]^{k+1}$. We say that the pair (G_1, G_2) is *weakly allowable* if for every $s_2 \in G_2$ there exists $s_1 \in G_1$ with s_1, s_2 being allowable.
- (c) A finite sequence (G_0, \dots, G_l) of subsets of $[\mathbb{N}]^{k+1}$ is said to be a *weakly allowable path* if for every $0 \leq i < l$ the pair (G_i, G_{i+1}) is weakly allowable.

We will also need a real function on two variables defined as follows.

Definition 8.6. Let

$$D = \left\{ (\varepsilon, \delta) \in [0, \frac{1}{2} - \frac{1}{4}\sqrt{2}] \times [0, 1] : (1 - 2\varepsilon)^2 + (1 - 2\varepsilon - \delta)^2 \geq 1 \right\}$$

and $h : D \rightarrow \mathbb{R}$ defined by the rule

$$h(\varepsilon, \delta) = (1 - (1 - 2\varepsilon - (1 - (1 - 2\varepsilon - \delta)^2)^{\frac{1}{2}})^2)^{\frac{1}{2}}$$

Before passing to the proof of Proposition 8.3, let us make some comments concerning the function $h(\varepsilon, \delta)$. First let us note that the curve

$$\mathcal{E} = \{(\varepsilon, \delta) \in \mathbb{R}^2 : (1 - 2\varepsilon)^2 + (1 - 2\varepsilon - \delta)^2 = 1\}$$

is an ellipse, since its image through the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined by

$$T(\varepsilon, \delta) = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}$$

is a circle centered at $(1, 1)$ and of radius 1. Moreover notice that $(\frac{1}{2} - \frac{1}{4}\sqrt{2}, 0)$ is the first intersection point of the curve \mathcal{E} and the ε -axis. Also the point $(0, 1)$ belongs to \mathcal{E} and the δ -axis is the tangent of \mathcal{E} at $(0, 1)$. Therefore the set D is a curved triangle with edges J_1, J_2, J_3 , where J_1 (respectively J_2) is the segment with endpoints $(\frac{1}{2} - \frac{1}{4}\sqrt{2}, 0)$ and $(0, 0)$ (respectively $(0, 0)$ and $(0, 1)$) and J_3 is the arc of \mathcal{E} which joins the $(0, 1)$ with $(\frac{1}{2} - \frac{1}{4}\sqrt{2}, 0)$.

It is easy to see that the function h is well defined on D . Moreover h is strictly increasing on D in the following sense: for all $(\varepsilon', \delta'), (\varepsilon, \delta) \in D$, with either $0 \leq \varepsilon' < \varepsilon$ and $0 \leq \delta' \leq \delta$ or $0 \leq \varepsilon' \leq \varepsilon$ and $0 \leq \delta' < \delta$, we have that $h(\varepsilon', \delta') < h(\varepsilon, \delta)$. Finally $h[D] = [0, 1]$, $h^{-1}(\{0\}) = \{(0, 0)\}$ and $h^{-1}(\{1\}) = J_3$.

Under the above we have the following.

Lemma 8.7. Let $(\varepsilon, \delta) \in D$. Let $x_1, x_2 \in \mathfrak{X}_{k+1}$ with disjoint finite supports such that $\|x_1\|, \|x_2\| \leq 1$ and $\|x_1 + x_2\| > 2 - 2\varepsilon$. Let $G_1 \subseteq \text{supp}(x_1)$ such that $\|x_1|_{G_1^c}\| \leq \delta$. Then there exists $G_2 \subseteq \text{supp}(x_2)$ satisfying the following.

- (a) The pair (G_1, G_2) is weakly allowable.
- (b) $\|x_2|_{G_2^c}\| \leq h(\varepsilon, \delta)$.

PROOF. Since $\|x_1 + x_2\| > 2 - 2\varepsilon$, there exists $\varphi \in W_{k+1}$ such that $\varphi(x_1 + x_2) > 2 - 2\varepsilon$. Since $\|x_1\|, \|x_2\| \leq 1$, we get that $\varphi(x_1) > 1 - 2\varepsilon$ and $\varphi(x_2) > 1 - 2\varepsilon$. The functional φ is of the form $\sum_{i=1}^n \lambda_i f_i$, where f_1, \dots, f_n are pairwise disjoint supported elements of \mathcal{G}_{k+1} . We set $I = \{1, \dots, n\}$ and we split it to I_1 and I_2 as follows:

$$I_1 = \{i \in I : \text{supp}(f_i) \cap G_1 \neq \emptyset\} \text{ and } I_2 = I \setminus I_1 = \{i \in I : \text{supp}(f_i) \subseteq G_1^c\}$$

We also set $\varphi_1 = \sum_{i \in I_1} \lambda_i f_i$ and $\varphi_2 = \sum_{i \in I_2} \lambda_i f_i$. Hence $\varphi(x_1) \leq \|x_1|_{G_1^c}\| \leq \delta$ and therefore $\varphi(x_1) > 1 - 2\varepsilon - \delta$. Applying Cauchy-Schwartz's inequality we get that

$$1 - 2\varepsilon - \delta < \varphi_1(x_1) = \sum_{i \in I_1} \lambda_i f_i(x_1) \leq \left(\sum_{i \in I_1} \lambda_i^2 \right)^{\frac{1}{2}} \left(\sum_{i \in I_1} f_i(x_1)^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i \in I_1} \lambda_i^2 \right)^{\frac{1}{2}}$$

where the last inequality holds since $\sum_{i \in I_1} \frac{f_i(x_1)}{(\sum_{j \in I_1} f_j(x_1)^2)^{\frac{1}{2}}} f_i$ belongs to W_{k+1} and $x_1 \leq 1$. Since $\sum_{i \in I} \lambda_i^2 \leq 1$, we have that

$$\left(\sum_{i \in I_2} \lambda_i^2 \right)^{\frac{1}{2}} < (1 - (1 - 2\varepsilon - \delta)^2)^{\frac{1}{2}}$$

Therefore

$$\varphi_2(x_2) = \sum_{i \in I_2} \lambda_i f_i(x_2) \leq \left(\sum_{i \in I_2} \lambda_i^2 \right)^{\frac{1}{2}} \left(\sum_{i \in I_2} f_i(x_2)^2 \right)^{\frac{1}{2}} < (1 - (1 - 2\varepsilon - \delta)^2)^{\frac{1}{2}}$$

Hence $\varphi_1(x_2) > 1 - 2\varepsilon - (1 - (1 - 2\varepsilon - \delta)^2)^{\frac{1}{2}}$. We set

$$G_2 = \text{supp}(x_2) \cap \text{supp}(\varphi_1)$$

Then by the definition of I_1 it is immediate that the pair (G_1, G_2) is weakly allowable. Finally, since $\|x_2|_{G_2}\|^2 + \|x_2|_{G_2^c}\|^2 \leq \|x_2\|^2 \leq 1$ and $\|x_2|_{G_2}\| \geq \varphi(x_2) = \varphi_1(x_2)$, we get that

$$\|x_2|_{G_2^c}\| \leq (1 - (1 - 2\varepsilon - (1 - (1 - 2\varepsilon - \delta)^2)^{\frac{1}{2}})^2)^{\frac{1}{2}} = h(\varepsilon, \delta)$$

□

Lemma 8.8. Let $m \in \mathbb{N}$. Then for every $\delta' \in (0, 1)$ there exists $\varepsilon' > 0$ with the following property. For every $0 < \varepsilon \leq \varepsilon'$, every sequence $(x_i)_{i=0}^m$ of disjointly and finitely supported vectors in \mathfrak{X}_{k+1} with $\|x_i\| \leq 1$, for all $0 \leq i \leq m$, if $\|x_i + x_{i+1}\| > 2 - 2\varepsilon$, for all $0 \leq i < m$, then there exists a weakly allowable path $(G_i)_{i=0}^m$ such that $G_i \subseteq \text{supp } x_i$ and $\|x_i|_{G_i^c}\| < \delta'$, for all $0 \leq i \leq m$.

PROOF. Let $m = 1$ and $\delta' \in (0, 1)$. Since $h(0, 0) = 0$, by the continuity of h there exists $\varepsilon' > 0$ such that for all $0 < \varepsilon \leq \varepsilon'$, we have that $(\varepsilon, 0) \in D$ and $h(\varepsilon, 0) < \delta'$. Let $0 < \varepsilon \leq \varepsilon'$ and x_0, x_1 with $\|x_0\| \leq 1$, $\|x_1\| \leq 1$ and $\|x_0 + x_1\| > 2 - 2\varepsilon$. We set $G_0 = \text{supp } x_0$. Then by Lemma 8.7, there exists $G_1 \subseteq \text{supp } x_1$ such that $\|x_1|_{G_1^c}\| \leq h(\varepsilon, 0) < \delta'$ and the result follows.

Suppose that the lemma holds for some $m \in \mathbb{N}$. Let $\delta' \in (0, 1)$. Since $h(0, 0) = 0$, by the continuity of h there exist $\varepsilon_1 > 0$ and $0 < \delta'_1 < \delta'$ such that for all $0 < \varepsilon \leq \varepsilon'_1$, we have that $(\varepsilon, \delta'_1) \in D$ and $h(\varepsilon, \delta'_1) < \delta'$. Let $\varepsilon'_2 > 0$ satisfying the inductive assumption for m and δ'_1 . Let $\varepsilon' = \min\{\varepsilon'_1, \varepsilon'_2\}$. Let $0 < \varepsilon \leq \varepsilon'$, $(x_i)_{i=0}^{m+1}$ be a sequence of disjointly and finitely supported vectors in \mathfrak{X}_{k+1} with $\|x_i\| \leq 1$, for all $0 \leq i \leq m$. Suppose that $\|x_i + x_{i+1}\| > 2 - 2\varepsilon$, for all $0 \leq i < m$. Then by the inductive assumption there exists a weakly allowable path $(G_i)_{i=0}^m$ such that $G_i \subseteq \text{supp } x_i$ and $\|x_i|_{G_i^c}\| < \delta'_1 < \delta'$, for all $0 \leq i \leq m$. By Lemma 8.7 there

exists $G_{m+1} \subseteq \text{supp} x_{m+1}$ such that $\|x_{m+1}|_{G_{m+1}^c}\| < h(\varepsilon, \delta'_1) < \delta'$ and G_m, G_{m+1} are weakly allowable. Hence $(G_i)_{i=0}^{m+1}$ is a weakly allowable path and $\|x_i|_{G_i^c}\| < \delta'$, for all $0 \leq i \leq m+1$. By induction the proof of the lemma is complete. \square

Lemma 8.9. Let $(G_i)_{i=0}^k$ be a weakly allowable path in $[\mathbb{N}]^{k+1}$ and $N = \max\{\max s : s \in G_0\}$. Then for every $0 \leq i \leq k$ and $s \in G_i$, $s(1) \leq N$.

PROOF. Using induction on $0 \leq i \leq k$ we will show that for every $s \in G_i$, $s(k+1-i) \leq N$, which easily yields the conclusion. By the definition of N , it is obvious that for every $t \in G_0$, $s(k+1) \leq N$. Suppose that for some $0 \leq i < m$ we have that $s(k+1-i) \leq N$ for all $s \in G_i$. Since the pair (G_i, G_{i+1}) is weakly allowable, we have that for every $s \in G_{i+1}$ there exists $s' \in G_i$ such that the set $\{s', s\}$ is allowable. Therefore $s(k+1-(i+1)) < s'(k+1-i) \leq N$. \square

Lemma 8.10. Let $f \in W_{k+1}^0$ and $x \in \mathfrak{X}_{k+1}$ of finite support. Then either

- (i) $\text{supp}(f) \cap \text{supp}(x) = \emptyset$, or
- (ii) $|\text{supp}(f)| \leq n_0^{k+1}$, where $n_0 = \max\{s(1) : s \in \text{supp}(x)\}$.

PROOF. Let $f \in W_{k+1}^0$. Then there exists $F_1 < \dots < F_{k+1}$ subsets of \mathbb{N} such that $\text{supp}(f) \subseteq F_1 \times \dots \times F_{k+1}$ and $|F_1| \leq \min F_1$. Hence $|\text{supp}(f)| \leq \min F_1$. Let $x \in \mathfrak{X}_{k+1}$ be finitely supported such that $\text{supp}(f) \cap \text{supp}(x) \neq \emptyset$. Let $s \in \text{supp}(f) \cap \text{supp}(x)$. Then $s(1) \geq \min F_1$ and $n_0 \geq s(1)$, where $n_0 = \max\{s(1) : s \in \text{supp}(x)\}$. Hence $n_0 \geq \min F_1$ and therefore $|\text{supp}(f)| \leq n_0^{k+1}$. \square

We are now ready for the proof of Proposition 8.3.

PROOF OF PROPOSITION 8.3. Let $\delta' = 0.01$ and choose $0 < \varepsilon < \delta$ satisfying the conclusion of Lemma 8.8 for $\delta' = 0.01$ and $m = k$. Assume on the contrary that the space \mathfrak{X}_{k+1} admits a plegma disjointly generated ℓ^1 spreading model of order k . By Proposition 6.1 and Remark 6.2 there exists a sequence $(x_t)_{t \in [\mathbb{N}]^k}$ in the unit ball of \mathfrak{X}_{k+1} which plegma disjointly generates an ℓ^1 spreading model of lower constant greater than $1 - \varepsilon$.

We fix for the following the set $t_0 = \{2, 4, \dots, 2k\}$. Let $t \in [\mathbb{N}]^k \upharpoonright \mathbb{N}$ with $t_0 < t$. By Proposition 1.17 there exists a plegma path $(t_j)_{j=0}^k$ in $[\mathbb{N}]^k$ (which depends on the choice of t), from t_0 to t of length k . By Lemma 8.8 there exists a weakly allowable path $(G_i)_{i=0}^k$ such that $G_i \subseteq \text{supp } x_{t_i}$ and $\|x_{t_i}|_{(G_i)^c}\| < \delta$, for all $i = 0, \dots, k$. We set

$$G_t = G_k, \quad x_t^1 = x_t|_{G_t} \quad \text{and} \quad x_t^2 = x_t - x_t^1$$

Hence for every $l \in \mathbb{N}$ and every plegma l -tuple $(t_j)_{j=1}^l$ in $[\mathbb{N}]^k$ with $s_1(1) > \max\{\max s_0, l-1\}$, we have that

$$(12) \quad \left\| \frac{1}{l} \sum_{i=1}^l x_{t_i}^1 \right\| > 1 - \varepsilon - \delta$$

Let $N_0 = \max\{\max s : s \in G_0\}$. Then by Lemma 8.9, we have that $s(1) \leq N_0$ for every $s \in \text{supp}(x_t^1) = G_t$ and $t \in [\mathbb{N}]^k \upharpoonright \mathbb{N}$ with $t_0 < t$.

Let $d = \lceil \frac{N_0^{k+1}}{\varepsilon^2} \rceil$ and $l_0 \in \mathbb{N}$ such that $\frac{d}{l_0} < \varepsilon$. Let $l \in \mathbb{N}$ and $(t_j)_{j=1}^l$ be a plegma l -tuple in $[\mathbb{N}]^k$ with $t_1(1) > \max\{\max t_0, l-1\}$. We will show that

$$\left\| \frac{1}{l_0} \sum_{j=1}^{l_0} x_{t_j}^1 \right\| < 2\varepsilon$$

which contradicts (12) and completes the proof.

Indeed, let $\varphi \in W_{k+1}$, $q \in \mathbb{N}$, $\lambda_1, \dots, \lambda_q \in \mathbb{R}$ with $\sum_{p=1}^q \lambda_p^2 \leq 1$ and $f_1, \dots, f_q \in W_{k+1}^0$ pairwise disjointly supported such that $\varphi = \sum_{p=1}^q \lambda_p f_p$. For every $j = 1, \dots, l_0$ we set

$$I_j = \left\{ p \in \{1, \dots, q\} : \text{supp}(f_p) \cap \text{supp}(x_{t_j}^1) \neq \emptyset \right\}$$

We also set

$$F_1 = \left\{ j \in \{1, \dots, l_0\} : \left(\sum_{p \in I_j} \lambda_p^2 \right)^{\frac{1}{2}} < \varepsilon \right\} \text{ and } F_2 = \{1, \dots, m\} \setminus F_1$$

So we have that

$$\begin{aligned} \varphi\left(\frac{1}{l_0} \sum_{j=1}^{l_0} x_{t_j}^1\right) &= \sum_{p=1}^q \lambda_p f_p\left(\frac{1}{l_0} \sum_{j=1}^{l_0} x_{t_j}^1\right) = \frac{1}{l_0} \sum_{j=1}^{l_0} \sum_{p=1}^q \lambda_p f_p(x_{t_j}^1) \\ &= \frac{1}{l_0} \sum_{j=1}^{l_0} \sum_{p \in I_j} \lambda_p f_p(x_{t_j}^1) \leq \frac{1}{l_0} \sum_{j=1}^{l_0} \left(\sum_{p \in I_j} \lambda_p^2 \right)^{\frac{1}{2}} \left(\sum_{p \in I_j} f_p(x_{t_j}^1)^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{l_0} \sum_{j \in F_1} \left(\sum_{p \in I_j} \lambda_p^2 \right)^{\frac{1}{2}} + \frac{1}{l_0} \sum_{j \in F_2} \left(\sum_{p \in I_j} \lambda_p^2 \right)^{\frac{1}{2}} \\ &\leq \frac{\varepsilon |F_1|}{l_0} + \frac{|F_2|}{l_0} \leq \varepsilon + \frac{|F_2|}{l_0} \end{aligned}$$

Since $(x_{t_j}^1)_{j=1}^{l_0}$ are disjointly supported, by Lemma 8.10, we have that for every $p = 1, \dots, q$, $|A_p| \leq N_0^{k+1}$, where

$$A_p = \left\{ j \in \{1, \dots, l_0\} : \text{supp}(f_p) \cap \text{supp}(x_{t_j}^1) \neq \emptyset \right\}$$

Hence

$$\sum_{j=1}^{l_0} \sum_{p \in I_j} \lambda_p^2 = \sum_{p=1}^q \sum_{j \in A_p} \lambda_p^2 \leq N_0^{k+1} \sum_{p=1}^q \lambda_p^2 \leq N_0^{k+1}$$

which yields that $|F_2| \leq d$. Therefore $\varphi\left(\frac{1}{l_0} \sum_{j=1}^{l_0} x_{t_j}^1\right) \leq \varepsilon + \frac{d}{l_0} < 2\varepsilon$. \square

2. Spaces admitting ℓ^1 as ξ -order spreading model but not less

In this section we show that for every countable ordinal ξ there exists a reflexive space \mathfrak{X}_ξ with an unconditional basis satisfying the following properties:

- (i) The space \mathfrak{X}_ξ admits ℓ^1 as a ξ -order spreading model.
- (ii) For every ordinal ζ such that $\zeta + 2 < \xi$, the space \mathfrak{X}_ξ does not admit ℓ^1 as a ζ -order spreading model.

Therefore, if ξ is a limit countable ordinal, then \mathfrak{X}_ξ is the minimum countable ordinal ζ such that $\mathcal{SM}_\zeta(\mathfrak{X}_\xi)$ contains a sequence equivalent to the usual basis of ℓ^1 .

2.1. The definition of the space \mathfrak{X}_ξ . Let $\xi < \omega_1$ and \mathcal{F} be a regular thin family of order ξ . We define the norm $\|\cdot\| : c_{00}(\mathcal{F}) \rightarrow \mathbb{R}$ by setting

$$\|x\| = \sup \left\{ \left(\sum_{i=1}^d \left(\sum_{j=1}^{l_i} |x(t_j^i)| \right)^2 \right)^{\frac{1}{2}} : d \in \mathbb{N}, (t_j^1)_{j=1}^{l_1}, \dots, (t_j^d)_{j=1}^{l_d} \in Plm(\mathcal{F}) \text{ and} \right. \\ \left. \text{for every } 1 \leq i_1 < i_2 \leq d, \right. \\ \left. \{t_j^{i_1} : 1 \leq j \leq l_{i_1}\} \cap \{t_j^{i_2} : 1 \leq j \leq l_{i_2}\} = \emptyset \right\}$$

for all $x \in c_{00}(\mathcal{F})$. We define $X_\xi = \overline{(c_{00}(\mathcal{F}), \|\cdot\|)}$. It is easy to see that a norming set for this space is the smallest $W \subseteq c_{00}(\mathcal{F})^\#$ such that the following are satisfied:

- (i) For every $d \in \mathbb{N}$, every plegma d -tuple $(t_j)_{j=1}^d$ in \mathcal{F} and $\varepsilon_1, \dots, \varepsilon_d \in \{0, 1\}$, the functional $f = \sum_{j=1}^d (-1)^{\varepsilon_j} e_{t_j}^*$ belongs to W and is called of type I.
- (ii) For every $d \in \mathbb{N}$, $f_1, \dots, f_d \in W$ of type I with disjoint supports and $a_1, \dots, a_d \in \mathbb{R}$ with $\sum_{j=1}^d a_j^2 \leq 1$, the functional $\varphi = \sum_{j=1}^d a_j f_j$ belongs to W and is called of type II.

It is immediate by the definition of the space X_ξ that its natural basis $(e_s)_{s \in \mathcal{F}}$ is unconditional. Our first aim is to prove that the space X_ξ does not contain any isomorphic copy of ℓ^1 . To this end we need the following notation and lemmas.

Notation 8.11. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of finite supported vectors in X_ξ . We say that $(x_n)_{n \in \mathbb{N}}$ is an \mathcal{F} -block sequence if for every $n \in \mathbb{N}$,

$$\max\{\max t : t \in \text{supp}(x_n)\} < \min\{\min t : t \in \text{supp}(x_{n+1})\}$$

The following lemma is immediate by the definition of the norm $\|\cdot\|$ and the observation that if $(x_n)_{n \in \mathbb{N}}$ is an \mathcal{F} -block sequence in X_ξ , then for every $n, m \in \mathbb{N}$, with $n \neq m$, there does not exist any plegma pair (s_1, s_2) , such that $s_1 \in \text{supp}(x_n)$ and $s_2 \in \text{supp}(x_m)$.

Lemma 8.12. Every seminormalized \mathcal{F} -block sequence in X_ξ is equivalent to the usual basis of ℓ^2 .

For every $l \in \mathbb{N}$ we recall that $\mathcal{F}_{[l]} = \{s \in \mathcal{F} : \min s = l\}$. We define $X_l = \overline{(e_s)_{s \in \mathcal{F}_{[l]}}}^{\|\cdot\|}$ and $P_l : X_\xi \rightarrow X_l$ such that for every $x \in c_{00}(\mathcal{F})$, $P_l(x) = \sum_{s \in \mathcal{F}_{[l]}} (x)_s e_s$. Since for every $l \in \mathbb{N}$ there does not exist any plegma pair in $\mathcal{F}_{[l]}$, the space X_l is isometric to ℓ^2 . Hence for every $l \in \mathbb{N}$ the space X_l is reflexive.

Proposition 8.13. Every subspace Z of X_ξ contains a further subspace W such that either there exists $l_0 \in \mathbb{N}$ such that $P_{l_0}|_W$ is an isomorphic embedding, or W is an isomorphic copy of ℓ^2 .

PROOF. Either there exists an $l_0 \in \mathbb{N}$ such that the operator $P_{l_0}|_Z : Z \rightarrow X_{l_0}$ is not strictly singular or for every $l \in \mathbb{N}$ the operator $P_l|_Z : Z \rightarrow X_l$ is strictly singular. In the first case it is immediate that there exists W infinite dimensional subspace of Z such that the operator $P_{l_0}|_W$ is an isomorphic embedding.

Suppose that the second case occurs. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive reals such that $\sum_{n=1}^\infty \varepsilon_n < \frac{1}{3}$. By induction we construct an \mathcal{F} -block sequence $(\tilde{w}_n)_{n \in \mathbb{N}}$ and a sequence $(w_n)_{n \in \mathbb{N}}$ in S_Z such that $\|w_n - \tilde{w}_n\| < \varepsilon_n$, for all $n \in \mathbb{N}$. Let $w_1 \in S_Z$ and $\tilde{w}_1 \in c_{00}(\mathcal{F})$ such that $\|w_1 - \tilde{w}_1\| < \varepsilon_1$. Suppose that $(w_i)_{i=1}^n$ and $(\tilde{w}_i)_{i=1}^n$ have been chosen. Let $l_0 = \max\{\max s : s \in \text{supp}(\tilde{w}_n)\}$. Since, for every

$1 \leq l \leq l_0$, the operator $P_l|_Z$ is strictly singular, there exists a subspace W of Z such that $\|P_l|_W\| < \frac{\varepsilon_{n+1}}{2l_0}$, for all $1 \leq l \leq l_0$. Let $w_{n+1} \in S_W$ and $w'_{n+1} = w_{n+1} - \sum_{l=1}^{l_0} P_l(w_{n+1})$. Pick also $\tilde{w}_{n+1} \in c_{00}(\mathbb{N}^k)$ such that $\text{supp}(\tilde{w}_{n+1}) \subseteq \text{supp}(w'_{n+1})$ and $\|w'_{n+1} - \tilde{w}_{n+1}\| < \frac{\varepsilon_{n+1}}{2}$. One can easily check that $\|w_{n+1} - \tilde{w}_{n+1}\| < \varepsilon_{n+1}$ and $\max\{\max s : s \in \text{supp}(\tilde{w}_n)\} < \min\{\min s : s \in \text{supp}(\tilde{w}_{n+1})\}$.

It is immediate that the sequence $(\tilde{w}_n)_{n \in \mathbb{N}}$ is 1-unconditional and seminormalized. By the choice of the sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ we have that the sequences $(\tilde{w}_n)_{n \in \mathbb{N}}$ and $(w_n)_{n \in \mathbb{N}}$ are equivalent. By Lemma 8.12 the sequence $(\tilde{w}_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^2 . Hence the subspace $\overline{\langle (w_n)_{n \in \mathbb{N}} \rangle}$ consists an isomorphic copy of ℓ^2 . \square

Corollary 8.14. The space X_ξ is ℓ^2 saturated.

Since the natural basis $(e_s)_{s \in \mathcal{F}}$ of the space X_ξ is unconditional, using James' theorem ([16]) and the above corollary we get the following.

Corollary 8.15. The space X_ξ is reflexive.

We close this section with the following proposition which will be used in the next subsection.

Proposition 8.16. Let \mathcal{G} be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{G}}$ a seminormalized \mathcal{G} -sequence in X_ξ satisfying the following:

- (i) The \mathcal{G} -subsequence is plegma disjointly supported.
- (ii) There exists $K \in \mathbb{N}$ such that $\max\{t(1) : t \in \text{supp}(x_s)\} \leq K$ for all $s \in \mathcal{G} \upharpoonright M$.

Then every \mathcal{G} -spreading model admitted by $(x_s)_{s \in \mathcal{G} \upharpoonright M}$ is equivalent to the usual basis of ℓ^2 .

PROOF. Let $c, C > 0$ such that $c \leq \|x_s\| \leq C$. Let also $L \in [M]^\infty$ such that the \mathcal{G} -subsequence $(x_s)_{s \in \mathcal{G} \upharpoonright L}$ generates a \mathcal{G} -spreading model. Let $n \in \mathbb{N}$, $(s_j)_{j=1}^n$ be a plegma n -tuple in $\mathcal{G} \upharpoonright L$ with $s_1(1) \geq L(n)$ and $a_1, \dots, a_n \in \mathbb{R}$. Since $(a_j \cdot x_{s_j})_{j=1}^n$ are disjoint supported, we have that

$$\left\| \sum_{j=1}^n a_j x_{s_j} \right\| \geq \left(\sum_{j=1}^n \|a_j x_{s_j}\|^2 \right)^{\frac{1}{2}} \geq c \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}}$$

We will complete the proof by showing that

$$\left\| \sum_{j=1}^n a_j x_{s_j} \right\| \leq CK^{\frac{1}{2}} \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}}$$

Indeed, let $\varphi \in W$. Then there exist $d \in \mathbb{N}$, $f_1, \dots, f_d \in W$ of type I and $b_1, \dots, b_d \in \mathbb{R}$, with $\sum_{q=1}^d b_q^2 \leq 1$, such that $\varphi = \sum_{q=1}^d b_q f_q$. For every $1 \leq q \leq d$ we set

$$E_q = \{j \in \{1, \dots, n\} : \text{supp}(f_q) \cap \text{supp}(x_{s_j}) \neq \emptyset\}$$

It is easy to check that for every $1 \leq q \leq d$ we have that $|E_q| \leq K$.

$$\begin{aligned}
\left| \varphi \left(\sum_{j=1}^n a_j x_{s_j} \right) \right| &\leq \sum_{q=1}^d \sum_{j=1}^n |b_q a_j f_q(x_{s_j})| = \sum_{q=1}^d \sum_{j \in E_q} |b_q a_j f_q(x_{s_j})| \\
&\leq \left(\sum_{q=1}^d \sum_{j \in E_q} |b_q|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{q=1}^d \sum_{j \in E_q} |a_j f_q(x_{s_j})|^2 \right)^{\frac{1}{2}} \\
&\leq K^{\frac{1}{2}} \cdot \left(\sum_{j=1}^n |a_j|^2 \sum_{q=1}^d |f_q(x_{s_j})|^2 \right)^{\frac{1}{2}} \leq K^{\frac{1}{2}} \cdot C \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

□

2.2. ℓ^1 spreading models of \mathfrak{X}_ξ . In this subsection we study the ℓ^1 spreading models of the space \mathfrak{X}_ξ . It is a direct consequence of the definition of the norm of the space that the natural basis $(e_s)_{s \in \mathcal{F}}$ generates the usual basis of ℓ^1 as an \mathcal{F} -spreading model. The main aim of this subsection is to show that \mathfrak{X}_ξ does not admit ℓ^1 as ζ -order spreading model for any $\zeta < \omega_1$ such that $\zeta + 2 < \xi$. The proof of this result goes as follows. Let \mathcal{G} be a regular thin family with $o(\mathcal{G}) < \xi$ and $(x_s)_{s \in \mathcal{G}}$ a bounded \mathcal{G} -sequence of finitely supported elements of \mathfrak{X}_ξ . We consider the next two cases. In the first one we assume that for every $s \in \mathcal{G}$ and $t \in \text{supp}(x_s)$, $|s| < |t|$. Then under the additional assumption that $(x_s)_{s \in \mathcal{G}}$ is plegma disjointly supported, we show that $(x_s)_{s \in \mathcal{G}}$ does not admit ℓ^1 as a \mathcal{G} -spreading model. The second case is the complemented one. Namely we assume that for every $s \in \mathcal{G}$ and $t \in \text{supp}(x_s)$, $|t| \leq |s|$ and again we show that $(x_s)_{s \in \mathcal{G}}$ does not admit ℓ^1 as a \mathcal{G} -spreading model. The final result follows from the above two cases.

Let us point out that a similar method was used in the proof of Theorem 1.23. To some extent Theorem 1.23 can be viewed as the set theoretical analogue of the present result.

2.2.1. Case I. The following lemma is similar to Lemma 8.7. For the sequel let $h : D \rightarrow \mathbb{R}$ be the function defined in Definition 8.6.

Lemma 8.17. Let $(\varepsilon, \delta) \in D$, with $\varepsilon > 0$. Let also $x_1, x_2 \in B_{X_\xi}$ with disjoint finite supports satisfying the following:

- (a) $\|x_1 + x_2\| > 2 - 2\varepsilon$ and
- (b) For every $t_1 \in \text{supp}(x_1)$ and $t_2 \in \text{supp}(x_2)$ the pair (t_2, t_1) is not plegma.

Then for every $G_1 \subseteq \text{supp}(x_1)$ such that $\|x_1|_{G_1^\varepsilon}\| \leq \delta$ there exists $G_2 \subseteq \text{supp}(x_2)$ such that

- (i) $\|x_2|_{G_2^\varepsilon}\| \leq h(\varepsilon, \delta)$.
- (ii) For every $t_2 \in G_2$ there exists $t_1 \in G_1$ such that the pair (t_1, t_2) is plegma.

PROOF. Let $\varphi \in W$ such that $\varphi(x_1 + x_2) > 2 - 2\varepsilon$. Then there exist $d \in \mathbb{N}$, $f_1, \dots, f_d \in W$ of type I with disjoint supports and $a_1, \dots, a_d \in \mathbb{R}$ with $\sum_{q=1}^d a_q^2 \leq 1$ such that $\varphi = \sum_{q=1}^d a_q f_q$. It is immediate that $\varphi(x_1), \varphi(x_2) > 1 - 2\varepsilon$. Let $A_1 = \{q \in \{1, \dots, d\} : \text{supp}(f_q) \cap G_1 \neq \emptyset\}$ and $A_2 = \{1, \dots, d\} \setminus A_1$. We define $\varphi_1 = \sum_{q \in A_1} a_q f_q$ and $\varphi_2 = \varphi - \varphi_1 = \sum_{q \in A_2} a_q f_q$. Let $G_2 = \text{supp}(x_2) \cap \text{supp}(\varphi_1)$. It is easy to see that (ii) is satisfied and $\varphi(x_2|_{G_2}) = \varphi_1(x_2|_{G_2}) = \varphi_1(x_2)$. Notice

that

$$\begin{aligned} 1 - 2\varepsilon - \delta &< \varphi(x_1|_{G_1}) = \varphi_1(x_1|_{G_1}) = \sum_{q \in A_1} a_q f_q(x_1|_{G_1}) \\ &\leq \left(\sum_{q \in A_1} a_q^2 \right)^{\frac{1}{2}} \left(\sum_{q \in A_1} f_q(x_1|_{G_1})^2 \right)^{\frac{1}{2}} \leq \left(\sum_{q \in A_1} a_q^2 \right)^{\frac{1}{2}} \end{aligned}$$

Since $\sum_{q=1}^d a_q^2 \leq 1$, we get that

$$\sum_{q \in A_2} a_q^2 < 1 - (1 - 2\varepsilon - \delta)^2$$

Hence

$$|\varphi_2(x_2)| \leq \sum_{q \in A_2} |a_q f_q(x_2)| \leq \left(\sum_{q \in A_2} |a_q|^2 \right)^{\frac{1}{2}} \left(\sum_{q \in A_2} |f_q(x_2)|^2 \right)^{\frac{1}{2}} < (1 - (1 - 2\varepsilon - \delta)^2)^{\frac{1}{2}}$$

Thus

$$\|x_2|_{G_2}\| \geq \varphi(x_2|_{G_2}) = \varphi_1(x_2) = \varphi(x_2) - \varphi_2(x_2) > 1 - 2\varepsilon - (1 - (1 - 2\varepsilon - \delta)^2)^{\frac{1}{2}}$$

By the definition of the space X_ξ we have that $\|x_2\|^2 \geq \|x_2|_{G_2}\|^2 + \|x_2|_{G_2^c}\|^2$. Hence $\|x_2|_{G_2^c}\| \leq (1 - (1 - 2\varepsilon - (1 - (1 - 2\varepsilon - \delta)^2)^{\frac{1}{2}})^2)^{\frac{1}{2}} = h(\varepsilon, \delta)$. \square

The proof of the following lemma is similar to the one of the previous lemma.

Lemma 8.18. Let $\varepsilon, \delta, x_1, x_2$ be as in Lemma 8.17. Then for every $G_2 \subseteq \text{supp}(x_2)$ such that $\|x_2|_{G_2^c}\| \leq \delta$ there exists $G_1 \subseteq \text{supp}(x_1)$ such that

- (i) $\|x_1|_{G_1^c}\| \leq h(\varepsilon, \delta)$.
- (ii) For every $t_1 \in G_1$ there exists $t_2 \in G_2$ such that the pair (t_1, t_2) is plegma.

Under the above lemmas we have the following.

Proposition 8.19. Let \mathcal{G} regular thin family, $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{G}}$ a \mathcal{G} -sequence in B_{X_ξ} satisfying the following

- (i) The \mathcal{G} -subsequence $(x_s)_{s \in \mathcal{G} \upharpoonright M}$ is plegma disjointly supported.
- (ii) For every plegma pair (s_1, s_2) in $\mathcal{G} \upharpoonright M$, every $t_1 \in \text{supp}(x_{s_1})$ and $t_2 \in \text{supp}(x_{s_2})$ the pair (t_2, t_1) is not plegma.
- (iii) For every $s \in \mathcal{G} \upharpoonright M$ and every $t \in \text{supp}(x_s)$ we have that $|t| > |s|$.

Then the \mathcal{G} -subsequence $(x_s)_{s \in \mathcal{G} \upharpoonright M}$ does not admit the usual basis of ℓ^1 as a \mathcal{G} -spreading model.

PROOF. Assume on the contrary that the \mathcal{G} -subsequence $(x_s)_{s \in \mathcal{G} \upharpoonright M}$ admits the usual basis of ℓ^1 as \mathcal{G} -spreading model. We inductively choose sequences $(\delta_n)_{n=0}^\infty$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ as follows. Let $\delta_0 = 0$ and pick $0 < \varepsilon_1 < \frac{1}{2} - \frac{1}{4}\sqrt{2}$. Then $(\varepsilon_1, \delta_0) \in D \setminus J_3$ and therefore $0 < h(\varepsilon_1, \delta_0) < 1$. We set $\delta_1 = h(\varepsilon_1, \delta_0)$. Suppose that $\varepsilon_1, \dots, \varepsilon_n$ and $\delta_0, \dots, \delta_n$ have been chosen such that for every $1 \leq k \leq n$

$$1 > \delta_k = h(\varepsilon_k, \delta_{k-1}) > 0$$

Then pick sufficiently small $\varepsilon_{n+1} > 0$ such that $(\varepsilon_{n+1}, \delta_n) \in D \setminus J_3$. Then we get $1 > h(\varepsilon_{n+1}, \delta_n) > 0$ and we set

$$\delta_{n+1} = h(\varepsilon_{n+1}, \delta_n)$$

It is clear that for every $n \in \mathbb{N}$ we have that $0 < \delta_n < 1$.

Let $L \in [M]^\infty$ such that the \mathcal{G} -subsequence $(x_s)_{s \in \mathcal{G} \upharpoonright L}$ generates the usual basis of ℓ^1 as \mathcal{G} -spreading model with respect to $(\varepsilon_n)_{n \in \mathbb{N}}$. We assume passing to an

infinite subset of L , that \mathcal{G} is very large in L . Let $s_0 \in \mathcal{G}$ such that $s_0 \sqsubseteq L(2\mathbb{N})$. We set

$$K = \max \{ \max t : t \in \text{supp}(x_{s_0}) \}$$

Claim: Let $L_1 = \{m \in L(2\mathbb{N}) : m > \max s_0\}$. For every $s \in \mathcal{G} \upharpoonright L_1$ there exists $G_s \subseteq \text{supp}(x_s)$ such that $\|x_s|_{G_s^c}\| \leq \delta_{|s_0|}$ and for all $t \in G_s$, $t(1) < K$.

PROOF OF CLAIM. Let $s \in \mathcal{G} \upharpoonright L_1$. Then $s \in \mathcal{G} \upharpoonright L(2\mathbb{N})$ satisfying $\max s_0 < \min s$ and therefore by Proposition 1.17 there exists plegma path $(s_j)_{j=0}^{|s_0|}$ in $\mathcal{G} \upharpoonright L$ from s_0 to s . Since the \mathcal{G} -subsequence $(x_s)_{s \in \mathcal{G} \upharpoonright L}$ generates the usual basis of ℓ^1 as \mathcal{G} -spreading model with respect to $(\varepsilon_n)_{n \in \mathbb{N}}$, we have that for every $1 \leq j \leq |s_0|$,

$$\|x_{s_j} + x_{s_{j-1}}\| > 2 - 2\varepsilon_j$$

We set $G_0 = \text{supp}(x_{s_0})$. Using Lemma 8.17 inductively for $j = 1, \dots, |s_0|$, we obtain $G_1, \dots, G_{|s_0|}$ satisfying the following:

- (i) $G_j \subseteq \text{supp}(x_{s_j})$
- (ii) $\|x_{s_j}|_{G_j^c}\| \leq \delta_j$ and
- (iii) for every $t \in G_j$ there exists $t' \in G_{j-1}$ such that the pair (t', t) is plegma.

Hence for every $t \in G_{|s_0|}$ there exists plegma path $(t_j)_{j=0}^{|s_0|}$ of length $|s_0|$ such that $t_{|s_0|} = t$ and $t_j \in G_j$ for all $0 \leq j \leq |s_0|$. Since \mathcal{F} is regular thin we have that $|t_j| \geq |t_0|$, for all $1 \leq j \leq |s_0|$. Hence by assumption (iii), we have that $|t_j| > |s_0|$ for all $0 \leq j \leq |s_0|$. Therefore

$$t(1) = t_{|s_0|}(1) < t_{|s_0|-1}(2) < \dots < t_{|s_0|-j}(j+1) < \dots < t_0(|s_0|+1) \leq K$$

and the proof of the claim is complete. \square

For every $s \in \mathcal{G} \upharpoonright L_1$ we set $x_s^1 = x_s|_{G_s}$ and $x_s^2 = x_s - x_s^1$. We choose $L_2 \in [L_1]^\infty$ such that $(x_s^1)_{s \in \mathcal{G} \upharpoonright L_2}$ and $(x_s^2)_{s \in \mathcal{G} \upharpoonright L_2}$ generate $(e_n^1)_{n \in \mathbb{N}}$ and $(e_n^2)_{n \in \mathbb{N}}$ respectively as \mathcal{G} -spreading models. Then $(e_n^1)_{n \in \mathbb{N}}$ is either trivial or by Lemma 8.16 and the above claim is equivalent to the usual basis of ℓ^2 . Hence by Corollary 4.2 $(e_n^2)_{n \in \mathbb{N}}$ is the usual basis of ℓ^1 and thus $\|e_1^2\| = 1$. The latter consists a contradiction since $\|x_s^2\| = \|x_s|_{G_s^c}\| \leq \delta_{|s_0|} < 1$, for all $s \in \mathcal{G} \upharpoonright L_2$. \square

2.2.2. Case II.

Lemma 8.20. Let \mathcal{H} regular thin family with $o(\mathcal{H}) < o(\mathcal{F})$ and $M \in [\mathbb{N}]^\infty$. Then there is no map $\Phi : \mathcal{H} \upharpoonright M \rightarrow \mathcal{P}(\mathcal{F})$ satisfying for every $v \in \mathcal{H} \upharpoonright M$ the following:

- (i) $\Phi(v) \neq \emptyset$.
- (ii) $|t| \leq |v|$, for all $t \in \Phi(v)$.
- (iii) $v(i) \leq t(i)$, for all $t \in \Phi(v)$ and $1 \leq i \leq |t|$.

PROOF. Suppose on the contrary that there exists a map $\Phi : \mathcal{H} \upharpoonright M \rightarrow \mathcal{P}(\mathcal{F})$ satisfying (i)-(iii). By Proposition 1.6 there exists $L \in [M]^\infty$ such that $\mathcal{H} \upharpoonright L \sqsubset \mathcal{F} \upharpoonright L$. Let $v \in \mathcal{H} \upharpoonright L$ and $u \in \mathcal{F} \upharpoonright L$ such that $v \sqsubset u$. Let $t \in \Phi(v)$. Then $|t| < |u|$ and $u(i) = v(i) \leq t(i)$, for all $1 \leq i \leq |t|$. The latter contradicts that $u \in \mathcal{F}$. \square

Proposition 8.21. Let \mathcal{G} be a regular thin family with $o(\mathcal{G}) < \xi$, $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{G}}$ a \mathcal{G} -sequence in B_{X_ξ} satisfying the following:

- (i) For every $s \in \mathcal{G} \upharpoonright M$ the vector x_s is of finite support.
- (ii) For every plegma pair (s_1, s_2) and every $t_1 \in \text{supp}(x_{s_1})$, $t_2 \in \text{supp}(x_{s_2})$ the pair (t_2, t_1) is not plegma.
- (iii) For every $s \in \mathcal{G} \upharpoonright M$ and $t \in \text{supp}(x_s)$, we have that $|t| \leq |s|$.

(iv) For every $s \in \mathcal{G} \upharpoonright M$ and $t \in \text{supp}(x_s)$, if $\min s = M(k)$ then $t(1) \geq k$.

Then the \mathcal{G} -subsequence $(x_s)_{s \in \mathcal{G} \upharpoonright M}$ does not admit the usual basis of ℓ^1 as a \mathcal{G} -spreading model.

PROOF. Suppose on the contrary that the \mathcal{G} -subsequence $(x_s)_{s \in \mathcal{G} \upharpoonright M}$ admits the usual basis of ℓ^1 as a \mathcal{G} -spreading model. Let $\delta_0 = 0, 01$. Since the function h is continuous at $(0, 0)$ and $h(0, 0) = 0$ we can inductively construct strictly decreasing null sequences of reals $(\varepsilon_n)_{n \in \mathbb{N}}$ and $(\delta_n)_{n \in \mathbb{N}}$ such that $\delta_1 < \delta_0$ and $h(\varepsilon_n, \delta_n) < \delta_{n-1}$, for all $n \in \mathbb{N}$.

We pass to some $L \in [M]^\infty$ such that \mathcal{G} is very large in L and the \mathcal{G} -subsequence $(x_s)_{s \in \mathcal{G} \upharpoonright L}$ generates the usual basis of ℓ^1 as a \mathcal{G} -spreading model with respect to $(\varepsilon_n)_{n \in \mathbb{N}}$. By property (iv) we have that for every $s \in \mathcal{G} \upharpoonright L$ and $t \in \text{supp}(x_s)$, if $\min s = L(k)$ then $t(1) \geq k$. We set

$$\mathcal{H} = \{v \in [\mathbb{N}]^{<\infty} : L(v) \in \mathcal{G}\}$$

and for every $v \in \mathcal{H}$ we set $z_v = x_{L(v)}$. It is immediate that the \mathcal{H} -sequence $(z_v)_{v \in \mathcal{H}}$ generates the usual basis of ℓ^1 as an \mathcal{H} -spreading model with respect to $(\varepsilon_n)_{n \in \mathbb{N}}$ and for every $v \in \mathcal{H}$ we have that $v(1) \leq t(1)$, for all $t \in \text{supp}(z_v)$. Notice also that $o(\mathcal{H}) = o(\mathcal{G}) < \xi = o(\mathcal{F})$.

For every $v \in \mathcal{H} \upharpoonright 2\mathbb{N}$, we select a plegma path $(v_j)_{j=1}^{|v|}$ in \mathcal{H} such that $v_1 = v$ and

$$\{n-1 : n \in v_{j-1} \setminus \{\min v_{j-1}\}\} \subseteq v_j$$

for all $1 < j \leq |v|$. Hence $v_j(1) = v(j) - (j-1)$, for all $1 \leq j \leq |v|$. Notice also that for every $1 < j \leq |v|$ we have that $\|x_{v_j} + x_{v_{j-1}}\| > 2 - 2\varepsilon_j$. We set $G_{|v|} = \text{supp}(z_{v_{|v|}})$ and using for $j = |v|, \dots, 2$, Lemma 8.18 we may get $G_{|v|-1}, \dots, G_1$ satisfying the following:

- (a) $G_j \subseteq \text{supp}(x_{v_j})$, for all $1 \leq j \leq |v|$.
- (b) $\|x_{v_j}|_{G_j^c}\| < \delta_{j-1}$, for all $1 \leq j \leq |v|$.
- (c) For every $t \in G_j$ there exists $t' \in G_{j+1}$ such that the pair (t, t') is plegma.

Hence for every $t_1 \in G_1$ there exists a plegma path $(t_j)_{j=1}^{|v|}$ such that $t_j \in G_j$, for all $1 \leq j \leq |v|$. Thus for every $t_1 \in G_1$ we have that for every $1 \leq j \leq |t_1| \leq |v|$, $t_1(j) \geq t_j(1) + (j-1) \geq v_j(1) + (j-1) = v_1(j) = v(j)$. We set $G_v = G_1$, which by (b) is nonempty. Hence there exists a map $\Phi : \mathcal{H} \upharpoonright 2\mathbb{N} \rightarrow \mathcal{P}(\mathcal{F})$ satisfying for every $v \in \mathcal{H} \upharpoonright M$ the following:

- (i) $\Phi(v) \neq \emptyset$.
- (ii) $|t| \leq |v|$, for all $t \in \Phi(v)$.
- (iii) $v(i) \leq t(i)$, for all $t \in \Phi(v)$ and $1 \leq i \leq |t|$.

This contradicts Lemma 8.20. □

2.2.3. The main result.

Theorem 8.22. For every $\zeta < \omega_1$, with $\zeta + 2 < \xi$, the space X_ξ does not admit ℓ^1 as a ζ -order spreading model.

PROOF. Assume on the contrary that for some $\zeta < \omega_1$ and $\zeta + 2 < \xi$, the space X_ξ admits ℓ^1 as a ζ -order spreading model. We choose $\phi : \mathcal{F} \rightarrow \mathbb{N}$ to be an onto and 1-1 map such that for every $s_1, s_2 \in \mathcal{F}$, if $\max s_1 < \max s_2$, then $\phi(s_1) < \phi(s_2)$. For every $n \in \mathbb{N}$ we set $e_n = e_{\phi^{-1}(n)}$. It is easy to check that the space X_ξ satisfies the property \mathcal{P} given in Definition 6.13. Since X_ξ is reflexive, by Corollary 6.17 the space X_ξ admits ℓ^1 as a plegma block generated spreading model of order ζ .

By Corollary 5.17 there exist a regular thin family \mathcal{G}_1 of order $\zeta + 1$, $M_1 \in [M]^\infty$ and a \mathcal{G}_1 -sequence $(x_s)_{s \in \mathcal{G}_1}$ in X_ξ such that $(x_s)_{s \in \mathcal{G}_1 \upharpoonright M_1}$ plegma block generates the usual basis of ℓ^1 as a \mathcal{G}_1 -spreading model. We may also assume that $(x_s)_{s \in \mathcal{G}_1 \upharpoonright M_1}$ is normalized. For every $s \in \mathcal{G}_1 \upharpoonright M_1$ we define $G_s^1 = \{t \in \text{supp}(x_s) : |t| \leq |s|\}$, $G_s^2 = \{t \in \text{supp}(x_s) : |t| > |s|\}$, $x_s^1 = x_s|_{G_s^1}$ and $x_s^2 = x_s|_{G_s^2}$.

First we will show that $(x_s^1)_{s \in \mathcal{G}_1 \upharpoonright M_1}$ does not admit ℓ^1 as a \mathcal{G}_1 -spreading model. Indeed, assume on the contrary. Then there is $M_2 \in [M_1]^\infty$ such that $(x_s^1)_{s \in \mathcal{G}_1 \upharpoonright M_2}$ plegma block generates ℓ^1 as a \mathcal{G}_1 -spreading model. By Corollary 5.17 there exist $M_3 \in [M_2]^\infty$ and a \mathcal{G}_2 -sequence $(z_v)_{v \in \mathcal{G}_2}$, where $\mathcal{G}_2 = [\mathbb{N}]^1 \oplus \mathcal{G}_1$, which satisfy the following:

- (i) The \mathcal{G}_2 -subsequence $(z_v)_{v \in \mathcal{G}_2 \upharpoonright M_3}$ plegma block generates the usual basis of ℓ^1 as a \mathcal{G} -spreading model.
- (ii) For every $v \in \mathcal{G}_2 \upharpoonright M_3$ there exist $m \in \mathbb{N}$ and $s_1, \dots, s_m \in \mathcal{G}_1$ satisfying the following:
 - (a) $z_v \in \langle x_{s_1}^1, \dots, x_{s_m}^1 \rangle$
 - (b) $|s_j| < |v|$, for all $1 \leq j \leq m$.

We may also assume that the \mathcal{G}_2 -subsequence $(z_v)_{v \in \mathcal{G}_2 \upharpoonright M_3}$ is normalized. For every $k \in \mathbb{N}$ and $v \in \mathcal{G}_2 \upharpoonright M_3$, we define $F_v^k = \{t \in \text{supp}(z_v) : \min t < k\}$, $z_v^{1,k} = z_v|_{F_v^k}$ and $z_v^{2,k} = z_v - z_v^{1,k}$. By Proposition 8.16, for every $k \in \mathbb{N}$ the \mathcal{G}_2 -subsequence $(z_v^{1,k})_{v \in \mathcal{G}_2 \upharpoonright M_3}$ does not admit ℓ^1 as a \mathcal{G}_2 -spreading model. Using Corollary 4.2 we inductively construct a decreasing sequence $(M'_k)_{k \in \mathbb{N}}$ of infinite subsets of M_3 such that $(z_v^{2,k})_{v \in \mathcal{G}_2 \upharpoonright M'_k}$ plegma block generates the usual basis of ℓ^1 and $\|z_v^{1,k}\| < \frac{1}{k}$, for all $v \in \mathcal{G}_2 \upharpoonright M'_k$. We pick $M_4 \in [M_3]^\infty$ such that $M_4(k) \in M'_k$, for all $k \in \mathbb{N}$. For every $v \in \mathcal{G}_2 \upharpoonright M_4$, we define $w_v = z_v^{2,k_v}$, where $k_v \in \mathbb{N}$ satisfying $\min v = M_4(k_v)$. Since $\|w_v - z_v\| < \frac{1}{k_v}$, for all $v \in \mathcal{G}_2 \upharpoonright M_4$, it is easy to check that the \mathcal{G}_2 -subsequence $(w_v)_{v \in \mathcal{G}_2 \upharpoonright M_4}$ generates the usual basis of ℓ^1 as a \mathcal{G}_2 -spreading model, which contradicts Proposition 8.21.

Since the \mathcal{G}_1 -subsequence $(x_s^1)_{s \in \mathcal{G}_1 \upharpoonright M_1}$ does not admit ℓ^1 as a \mathcal{G}_1 -spreading model, by Corollary 4.2 we get that the \mathcal{G}_1 -subsequence $(x_s^2)_{s \in \mathcal{G}_1 \upharpoonright M_1}$ admits the usual basis of ℓ^1 as a \mathcal{G}_1 -spreading model, which contradicts Proposition 8.19. \square

CHAPTER 9

Strong k -order and k -order spreading models

In this chapter we construct two spaces, a non reflexive and a reflexive one, which show that for $k > 1$, the strong k -order spreading models are a distinct subclass of the k -order ones. We first present and study a general class of norms. The desired examples are special cases of that class.

1. The general construction

Let $k \in \mathbb{N}$, with $k > 1$. For every $l \in \mathbb{N}$ let $C_l = \{s \in [\mathbb{N}]^k : \min s = l\}$ and $P_l : c_{00}([\mathbb{N}]^k) \rightarrow c_{00}(C_l)$ defined by $P_l(x) = \sum_{s \in C_l} x(s)e_s$, for all $x \in c_{00}([\mathbb{N}]^k)$. Let $(\|\cdot\|_l)_{l \in \mathbb{N}}$ be a sequence of norms defined on $c_{00}(\mathbb{N})$ such that for every $l \in \mathbb{N}$ the following are satisfied:

- (i) The basis $(e_n)_{n \in \mathbb{N}}$ is 1-unconditional under the norm $\|\cdot\|_l$.
- (ii) For every $n \in \mathbb{N}$, $\|e_n\|_l = 1$.

By the unconditionality property, for every $l \in \mathbb{N}$, we consider the norm $\|\cdot\|_l$ be also defined on $c_{00}(C_l)$. Fix $1 < q < p < \infty$. We define the norm $\|\cdot\|_{q,p} : c_{00}([\mathbb{N}]^k) \rightarrow \mathbb{R}$ such that

$$\|x\|_{q,p} = \sup \left\{ \left(\sum_{i=1}^d \left(\sum_{j=1}^{m_i} |x(s_j^i)|^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} : d \in \mathbb{N}, m_i \in \mathbb{N}, (s_j^i)_{j=1}^{m_i} \text{ is plegma} \right. \\ \left. \text{in } [\mathbb{N}]^k \text{ for all } 1 \leq i \leq d \text{ and } \left\{ s_j^{i_1}(1) \right\}_{j=1}^{m_{i_1}} \cap \left\{ s_j^{i_2}(1) \right\}_{j=1}^{m_{i_2}} = \emptyset, \right. \\ \left. \text{for all } 1 \leq i_1 < i_2 \leq d \right\}$$

Let $\|\cdot\|_{(1)} : c_{00}([\mathbb{N}]^k) \rightarrow \mathbb{R}$ be the norm defined by

$$\|x\|_{(1)} = \left(\sum_{l=1}^{\infty} \|P_l(x)\|_l^p \right)^{\frac{1}{p}}$$

for all $x \in c_{00}([\mathbb{N}]^k)$. We also set $\|\cdot\|_{(2)} = \|\cdot\|_{q,p}$. Finally we define the norm $\|\cdot\| : c_{00}([\mathbb{N}]^k) \rightarrow \mathbb{R}$ by setting

$$\|x\| = \max\{\|x\|_{(1)}, \|x\|_{(2)}\}$$

for all $x \in c_{00}([\mathbb{N}]^k)$ and we define $X = \overline{(c_{00}([\mathbb{N}]^k), \|\cdot\|)}$. It is immediate that the sequence $(e_s)_{s \in [\mathbb{N}]^k}$ forms an 1-unconditional basis for the space X . Notice also that for every $l \in \mathbb{N}$ and $x \in c_{00}(C_l)$ we have that $\|x\| = \|x\|_l$. Hence the subspace $c_{00}(C_l)$ of X is isometric to $(c_{00}(\mathbb{N}), \|\cdot\|_l)$, for all $l \in \mathbb{N}$. For every $l \in \mathbb{N}$ we define $X_l = \overline{(c_{00}(C_l), \|\cdot\|_l)}$.

Notation 9.1. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $c_{00}([\mathbb{N}]^k)$. We will say that $(x_n)_{n \in \mathbb{N}}$ is $[\mathbb{N}]^k$ -block if for every $n_1 < n_2$ in \mathbb{N} and for every $s_1 \in \text{supp}(x_{n_1})$ and $s_2 \in \text{supp}(x_{n_2})$ we have that $\max s_2 < \min s_1$.

Remark 9.2. It is easy to see that if $(x_n)_{n \in \mathbb{N}}$ is a $[\mathbb{N}]^k$ -block sequence in X , then

$$\left\| \sum_{l=1}^r a_l x_l \right\|_{(i)} = \left(\sum_{l=1}^r |a_l|^p \cdot \|x_l\|_{(i)}^p \right)^{\frac{1}{p}}$$

for all $r \in \mathbb{N}$, $a_1, \dots, a_r \in \mathbb{R}$ and $i \in \{1, 2\}$.

Lemma 9.3. Every seminormalized $[\mathbb{N}]^k$ -block sequence in X is equivalent to the usual basis of ℓ^p .

PROOF. Let $(x_n)_{n \in \mathbb{N}}$ be a seminormalized $[\mathbb{N}]^k$ -block sequence in X and $c, C > 0$ such that $c \leq \|x_n\| \leq C$, for all $n \in \mathbb{N}$. Let $r \in \mathbb{R}$ and $a_1, \dots, a_r \in \mathbb{R}$. Then by Remark 9.2 for every $i \in \{1, 2\}$ we have that

$$\left\| \sum_{l=1}^r a_l x_l \right\|_{(i)} = \left(\sum_{l=1}^r |a_l|^p \cdot \|x_l\|_{(i)}^p \right)^{\frac{1}{p}} \leq \left(\sum_{l=1}^r |a_l|^p \cdot \|x_l\|^p \right)^{\frac{1}{p}} \leq C \left(\sum_{l=1}^r |a_l|^p \right)^{\frac{1}{p}}$$

Hence

$$\left\| \sum_{l=1}^r a_l x_l \right\| \leq C \left(\sum_{l=1}^r |a_l|^p \right)^{\frac{1}{p}}$$

Let $E_1 = \{l \in \{1, \dots, r\} : \|x_l\|_{(1)} \geq \|x_l\|_{(2)}\}$ and $E_2 = \{1, \dots, r\} \setminus E_1$. Let also $i_0 \in \{1, 2\}$ such that

$$\left(\sum_{l \in E_{i_0}} |a_l|^p \right)^{\frac{1}{p}} \geq \frac{1}{2^{\frac{1}{p}}} \left(\sum_{l=1}^r |a_l|^p \right)^{\frac{1}{p}}$$

Then by Remark 9.2, we have that

$$\begin{aligned} \left\| \sum_{l=1}^r a_l x_l \right\| &\geq \left\| \sum_{l=1}^r a_l x_l \right\|_{(i_0)} = \left(\sum_{l=1}^r |a_l|^p \cdot \|x_l\|_{(i_0)}^p \right)^{\frac{1}{p}} \\ &\geq \left(\sum_{l \in E_{i_0}} |a_l|^p \cdot \|x_l\|_{(i_0)}^p \right)^{\frac{1}{p}} = \left(\sum_{l \in E_{i_0}} |a_l|^p \cdot \|x_l\|^p \right)^{\frac{1}{p}} \\ &\geq c \left(\sum_{l \in E_{i_0}} |a_l|^p \right)^{\frac{1}{p}} \geq \frac{c}{2^{\frac{1}{p}}} \left(\sum_{l=1}^r |a_l|^p \right)^{\frac{1}{p}} \end{aligned}$$

□

The following corollary follows by a sliding hump argument and Lemma 9.3.

Corollary 9.4. Let $(x_n)_{n \in \mathbb{N}}$ be a seminormalized sequence in X such that $P_l(x_n) \rightarrow 0$, for all $l \in \mathbb{N}$. Then $(x_n)_{n \in \mathbb{N}}$ contains a subsequence equivalent to the usual basis of ℓ^p .

Proposition 9.5. Every subspace Z of X contains a further subspace W such that either there exists $l_0 \in \mathbb{N}$ such that $P_{l_0}|_W$ is an isomorphic embedding, or W is an isomorphic copy of ℓ^p .

PROOF. Either there exists an $l_0 \in \mathbb{N}$ such that the operator $P_{l_0}|_Z : Z \rightarrow X_{l_0}$ is not strictly singular or for every $l \in \mathbb{N}$ the operator $P_l|_Z : Z \rightarrow X_{l_0}$ is strictly singular. In the first case it is immediate that there exists W infinite dimensional subspace of Z such that the operator $P_{l_0}|_W$ is an isomorphic embedding.

Suppose that the second case occurs. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive reals such that $\sum_{n=1}^{\infty} \varepsilon_n < \frac{1}{3}$. By induction we construct an $[\mathbb{N}]^k$ -block sequence

$(\tilde{w}_n)_{n \in \mathbb{N}}$ and a sequence $(w_n)_{n \in \mathbb{N}}$ in S_Z such that $\|w_n - \tilde{w}_n\| < \varepsilon_n$, for all $n \in \mathbb{N}$. Let $w_1 \in S_Z$ and $\tilde{w}_1 \in c_{00}([\mathbb{N}]^k)$ such that $\|w_1 - \tilde{w}_1\| < \varepsilon_1$. Suppose that $(w_i)_{i=1}^n$ and $(\tilde{w}_i)_{i=1}^n$ have been chosen. Let $l_0 = \max\{\max s : s \in \text{supp}(\tilde{w}_n)\}$. Since, for every $1 \leq l \leq l_0$, the operator $P_l|_Z$ is strictly singular, there exists a subspace W of Z such that $\|P_l|_W\| < \frac{\varepsilon_{n+1}}{2l_0}$, for all $1 \leq l \leq l_0$. Let $w_{n+1} \in S_W$ and $w'_{n+1} = w_{n+1} - \sum_{l=1}^{l_0} P_l(w_{n+1})$. Pick also $\tilde{w}_{n+1} \in c_{00}([\mathbb{N}]^k)$ such that $\text{supp}(\tilde{w}_{n+1}) \subseteq \text{supp}(w'_{n+1})$ and $\|w'_{n+1} - \tilde{w}_{n+1}\| < \frac{\varepsilon_{n+1}}{2}$. One can easily check that $\|w_{n+1} - \tilde{w}_{n+1}\| < \varepsilon_{n+1}$ and $\max\{\max s : s \in \text{supp}(\tilde{w}_n)\} < \min\{\min s : s \in \text{supp}(\tilde{w}_{n+1})\}$.

It is immediate that the sequence $(\tilde{w}_n)_{n \in \mathbb{N}}$ is 1-unconditional and seminormalized. By the choice of the sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ we have that the sequences $(\tilde{w}_n)_{n \in \mathbb{N}}$ and $(w_n)_{n \in \mathbb{N}}$ are equivalent. By Lemma 9.3 the sequence $(\tilde{w}_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^p . Hence the subspace $\overline{(w_n)_{n \in \mathbb{N}}}$ consists an isomorphic copy of ℓ^p . \square

Corollary 9.6. The following are satisfied:

- (i) The space X contains an isomorphic copy of ℓ^r (resp. c_0), for every $r \neq p$, if and only if there exists $l \in \mathbb{N}$ such that the space X_l contains an isomorphic copy of ℓ^r (resp. c_0).
- (ii) The space X is reflexive if and only if, for each $l \in \mathbb{N}$, the space X_l is reflexive.

PROOF. (i) Let $r \in [1, \infty)$, with $r \neq p$. Suppose that X contains an isomorphic copy Z of ℓ^r . Then by Proposition 9.5 we conclude that there exists subspace W of Z such that either W is an isomorphic copy of ℓ^r or for some $l \in \mathbb{N}$, $P_l|_W$ is an isomorphic embedding of W into X_l . Since W is a subspace of Z , it contains a further subspace W' isomorphic to ℓ^r . Therefore the first alternative is impossible and we get that there exists an $l \in \mathbb{N}$ such that X_l contains an isomorphic copy of ℓ^r . Conversely, since for every $l \in \mathbb{N}$, X_l is a subspace of X , if X_l contains an isomorphic copy of ℓ^r , then so does X . The arguments concerning c_0 are identical.

(ii) If X is reflexive then the same holds for every subspace of X . In particular X_l is reflexive for all $l \in \mathbb{N}$. Conversely suppose that X_l is reflexive for every $l \in \mathbb{N}$. By (i) we have that X does not contain any isomorphic copy of ℓ^1 or c_0 . Since X has an unconditional basis, by James' theorem (c.f. [16]), we conclude that X is reflexive. \square

Lemma 9.7. Assume that the space X does not contain any isomorphic copy of c_0 . Let $(x_n)_{n \in \mathbb{N}}$ be a seminormalized Schauder basic sequence in X such that for every $l \in \mathbb{N}$ the sequence $(P_l(x_n))_{n \in \mathbb{N}}$ is norm convergent. Then for every $l \in \mathbb{N}$ the sequence $(P_l(x_n))_{n \in \mathbb{N}}$ is a null sequence and $(x_n)_{n \in \mathbb{N}}$ contains a subsequence equivalent to the usual basis of ℓ^p .

PROOF. For every $l \in \mathbb{N}$, let $y_l \in X_l$ be the norm limit of $(P_l(x_n))_{n \in \mathbb{N}}$. By Corollary 9.4 it suffices to show that $y_l = 0$ for all $l \in \mathbb{N}$. Since $(e_s)_{s \in [\mathbb{N}]^k}$ is unconditional and X does not contain any isomorphic copy of c_0 we get that $(e_s)_{s \in [\mathbb{N}]^k}$ is boundedly complete.

For every $l_0 \in \mathbb{N}$, we have

$$\sum_{l=1}^{l_0} P_l(x_n) \xrightarrow{n \rightarrow \infty} \sum_{l=1}^{l_0} y_l$$

Hence $(\|\sum_{l=1}^{l_0} y_l\|)_{l_0=1}^\infty$ is a bounded sequence and therefore there exists $y \in X$ such that $\sum_{l=1}^{l_0} y_l \xrightarrow{l_0 \rightarrow \infty} y$. Hence for each $l \in \mathbb{N}$, $P_l(y) = y_l$ and so we need to show that $y = 0$.

Suppose on the contrary that $y \neq 0$ and let $z_n = x_n - y$ for all $n \in \mathbb{N}$. Since $(x_n)_{n \in \mathbb{N}}$ is Schauder basic, we have that $(z_n)_{n \in \mathbb{N}}$ is not a null sequence. Notice that for every $l \in \mathbb{N}$, we have that $P_l(z_n) \xrightarrow{n \rightarrow \infty} 0$. By Corollary 9.4 we can pass to a subsequence $(z_{k_n})_{n \in \mathbb{N}}$ such that $(z_{k_n})_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^p . In particular we have that the sequence $(z_{k_n})_{n \in \mathbb{N}}$ is Cesàro summable to zero. Therefore there exists $n_0 > 0$ such that

$$\left\| \frac{1}{n_0} \sum_{n=1}^{n_0} z_{k_n} \right\| < \frac{\|y\|}{3C} \quad \text{and} \quad \left\| \frac{1}{n_0} \sum_{n=n_0+1}^{2n_0} z_{k_n} \right\| < \frac{\|y\|}{3C}$$

where C is the basis constant of $(x_n)_n$. This yields that

$$\frac{2\|y\|}{3} < \left\| \frac{1}{n_0} \sum_{n=1}^{n_0} x_{k_n} \right\| \leq C \left\| \frac{1}{n_0} \sum_{n=1}^{n_0} x_{k_n} - \frac{1}{n_0} \sum_{n=n_0+1}^{2n_0} x_{k_n} \right\| < \frac{2\|y\|}{3}$$

which is a contradiction. \square

Proposition 9.8. The space X admits ℓ^q as a spreading model of order k . In particular, the natural basis $(e_s)_{s \in [\mathbb{N}]^k}$ of X generates ℓ^q as an $[\mathbb{N}]^k$ -spreading model.

PROOF. Let $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}$ and $(s_j)_{j=1}^n$ be a plegma n -tuple in $[\mathbb{N}]^k$ with $s_1(1) \geq n$. It is immediate that

$$\left\| \sum_{j=1}^n a_j e_{s_j} \right\|_{(1)} = \left(\sum_{j=1}^n |a_j|^p \right)^{\frac{1}{p}}$$

Let F_1, \dots, F_d disjoint subsets of $\{1, \dots, n\}$. Since $1 < q < p$, we have that

$$\left(\sum_{i=1}^d \left(\sum_{j \in F_i} |a_j|^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^d \left(\sum_{j \in F_i} |a_j|^q \right)^{\frac{q}{q}} \right)^{\frac{1}{q}} = \left(\sum_{j=1}^n |a_j|^q \right)^{\frac{1}{q}}$$

the above imply that

$$\left\| \sum_{j=1}^n a_j e_{s_j} \right\| = \left\| \sum_{j=1}^n a_j e_{s_j} \right\|_{(2)} = \left(\sum_{j=1}^n |a_j|^q \right)^{\frac{1}{q}}$$

\square

2. The nonreflexive case

The main result of this section is the following.

Theorem 9.9. For every $1 < q < p < \infty$ and $k > 1$, there exists a Banach space $X_{1,p,q}^k$ with an unconditional basis such that every seminormalized Schauder basic sequence $(x_n)_{n \in \mathbb{N}}$ in $X_{1,p,q}^k$ contains a subsequence which is equivalent either to the usual basis of ℓ^1 or to the usual basis of ℓ^p . Moreover the space $X_{1,p,q}^k$ admits ℓ^q as a spreading model of order k .

PROOF. For $1 < q < p$, let $X_{1,q,p}$ be the space resulting from the construction of the previous section by setting for every $l \in \mathbb{N}$, $\|\cdot\|_l = \|\cdot\|_{\ell^1}$. Let $(x_n)_{n \in \mathbb{N}}$ be a seminormalized Schauder basic sequence in X . Then one of the following holds:

- (i) There exist $l_0 \in \mathbb{N}$ and $M_0 \in [\mathbb{N}]^\infty$ such that the sequence $(P_{l_0}(x_n))_{n \in M_0}$ does not contain any norm convergent subsequence.
- (ii) For every $l \in \mathbb{N}$ and $M \in [\mathbb{N}]^\infty$, the sequence $(P_l(x_n))_{n \in M}$ contains a norm convergent subsequence.

Suppose that (i) holds. Then $(P_{l_0}(x_n))_{n \in M_0}$ is a seminormalized sequence in ℓ^1 with no norm Cauchy subsequence. By the well known Rosenthal's ℓ^1 -theorem and the Schur property of ℓ^1 , we conclude that there exists $L \in [M_0]^\infty$ such that $(P_{l_0}(x_n))_{n \in L}$ is equivalent to the usual basis of ℓ^1 . Since the basis $(e_s)_{s \in [\mathbb{N}]^k}$ is unconditional, we get that the sequence $(x_n)_{n \in L}$ is also equivalent to the usual basis of ℓ^1 .

Suppose that (ii) holds. Then we may pass to an $M \in [\mathbb{N}]^\infty$ such that the sequence $(P_l(x_n))_{n \in M}$ converges. By Corollary 9.6, we have that $X_{1,p,q}^k$ does not contain any isomorphic copy of c_0 and by Lemma 9.7, we have that there exists a further subsequence of $(x_n)_{n \in M}$ equivalent to the usual basis of ℓ^p .

Finally, by Proposition 9.8 we have that the basis $(e_s)_{s \in [\mathbb{N}]^k}$ of the space $X_{1,p,q}^k$ generates ℓ^q as a k -order spreading model. \square

Corollary 9.10. The space $X_{1,q,p}^k$ does not admit ℓ^q as a strong spreading model of any order. Precisely, every strong spreading model of any order admitted by $X_{1,q,p}^k$ is equivalent either to the usual basis of ℓ^1 or to the usual basis of ℓ^p .

PROOF. Since for every $r \in [1, \infty)$ every strong spreading model (of order one) of ℓ^1 or ℓ^p is equivalent to the usual basis of ℓ^1 or ℓ^p , it suffices to show that every strong spreading model of order one admitted by X is either ℓ^1 or ℓ^p .

Indeed, let $(x_n)_{n \in \mathbb{N}}$ be a seminormalized Schauder basic sequence in X which generates a spreading model $(e_n)_{n \in \mathbb{N}}$. By Theorem 9.11 we have that $(x_n)_{n \in \mathbb{N}}$ contains a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ equivalent either to the usual basis of ℓ^1 or to the usual basis of ℓ^p . Since $(x_{k_n})_{n \in \mathbb{N}}$ also generates $(e_n)_{n \in \mathbb{N}}$, we have that $(e_n)_{n \in \mathbb{N}}$ is equivalent either to the usual basis of ℓ^1 or to the usual basis of ℓ^p . \square

3. The reflexive case

The main result of this section is the following.

Theorem 9.11. For every $1 < q < p < \infty$ and $k > 1$, there exists a reflexive space $X_{T,p,q}^k$ with an unconditional basis such that every seminormalized Schauder basic sequence $(x_n)_{n \in \mathbb{N}}$ in $X_{T,p,q}^k$ contains a subsequence which is either equivalent to the usual basis of ℓ^p or generates a spreading model (of order one) equivalent to the usual basis of ℓ^1 . Moreover the space $X_{T,p,q}^k$ admits ℓ^q as a spreading model of order k .

PROOF. For $1 < q < p$, let $X_{T,q,p}^k$ be the space resulting from the general construction by setting for every $l \in \mathbb{N}$, $\|\cdot\|_l = \|\cdot\|_T$, where $\|\cdot\|_T$ denotes the norm defined on Tsirelson's space. Since Tsirelson's space is reflexive, by Corollary 9.6 we get that $X_{T,q,p}^k$ is reflexive. Let $(x_n)_{n \in \mathbb{N}}$ be a seminormalized Schauder basic sequence in $X_{T,q,p}^k$. Since $X_{T,q,p}^k$ is reflexive and $(x_n)_{n \in \mathbb{N}}$ is Schauder basic, we get that $(x_n)_{n \in \mathbb{N}}$ is weakly null. By the standard sliding hump argument and by passing to a subsequence of $(x_n)_{n \in \mathbb{N}}$, we may suppose that the sequence $(x_n)_{n \in \mathbb{N}}$ is finitely disjointly supported. Observe that one of the following holds:

- (i) For every $l \in \mathbb{N}$, $P_l(x_n) \xrightarrow{n \rightarrow \infty} 0$.

- (ii) There exist $l_0 \in \mathbb{N}$, $\theta > 0$ and $M_0 \in [\mathbb{N}]^\infty$ such that $\|P_{l_0}(x_n)\| > \theta$ for all $n \in M_0$.

Suppose that (i) holds. By Corollary 9.6, we have that $X_{T,p,q}^k$ does not contain any isomorphic copy of c_0 and by Lemma 9.7, we have that there exists a further subsequence of $(x_n)_{n \in M_0}$ equivalent to the usual basis of ℓ^p .

Suppose that (ii) holds. Then the sequence $(P_{l_0}(x_n))_{n \in M_0}$ actually forms a seminormalized block sequence in Tsirelson's space. Since every spreading model generated by a seminormalized Schauder basic sequence in Tsirelson's space is equivalent to the usual basis of ℓ^1 , there exist $L \in [M_0]^\infty$ such that the sequence $(P_{l_0}(x_n))_{n \in L}$ generate ℓ^1 as spreading model. Since the basis $(e_s)_{s \in [\mathbb{N}]^k}$ is unconditional, we easily get that the subsequence $(x_n)_{n \in L}$ also generates ℓ^1 as spreading model.

Moreover, by Proposition 9.8 we have that the basis $(e_s)_{s \in [\mathbb{N}]^k}$ of the space $X_{T,p,q}^k$ generates ℓ^q as a k - order spreading model. \square

The proof of the following corollary is similar to the one of Corollary 9.10.

Corollary 9.12. The space $X_{T,q,p}^k$ does not admit ℓ^q as a strong spreading model of any order. Precisely, every strong spreading model of any order admitted by $X_{T,q,p}^k$ is equivalent either to the usual basis of ℓ^1 or to the usual basis of ℓ^p .

Spreading models do not occur everywhere as plegma block generated

In this chapter we construct a reflexive space X with an unconditional basis such that X admits ℓ^1 as a spreading model (of order ω) but not as a block generated spreading model (of any order). The space X does not satisfy the property \mathcal{P} (see Definition 6.13) and therefore the condition concerning this property in Theorem 6.15 is necessary.

1. The construction and the reflexivity of the space X

We recall that the Schreier family $\mathcal{S} = \{s \in [\mathbb{N}]^{<\omega} : |s| = \min s\}$ is a regular thin family of order ω . Let

$$\mathcal{H} = \{(t_j)_{j=1}^k : k \in \mathbb{N}, (t_j)_{j=1}^k \in Plm_k(\widehat{\mathcal{S}}) \text{ and } k-1 \leq |t_1| = \dots = |t_k|\}$$

Notice that for every $t \in \widehat{\mathcal{S}}$, we have that $(t) \in \mathcal{H}$. Moreover for every $(t_j)_{j=1}^k \in \mathcal{H}$, we have that the elements of the set $\{t_j : j = 1, \dots, k\}$ are pairwise \sqsubseteq -incomparable. Let $d \in \mathbb{N}$ and for every $1 \leq q \leq d$, let $k_d \in \mathbb{N}$ and $(t_j^q)_{j=1}^{k_q} \in \mathcal{H}$. We will say that $(t_j^1)_{j=1}^{k_1}, \dots, (t_j^d)_{j=1}^{k_d}$ are *incomparable* if the elements of the set $\cup_{q=1}^d \{t_j^q : 1 \leq j \leq k_q\}$ are pairwise \sqsubseteq -incomparable.

We define the following norm on $c_{00}(\widehat{\mathcal{S}})$:

$$\|x\| = \sup \left\{ \left(\sum_{q=1}^d \left(\sum_{j=1}^{k_q} |x(t_j^q)| \right)^2 \right)^{\frac{1}{2}} \right\}$$

where the supremum is taken over all $d \in \mathbb{N}$ and incomparable $(t_j^1)_{j=1}^{k_1}, \dots, (t_j^d)_{j=1}^{k_d}$ in \mathcal{H} . We set $X = \overline{c_{00}(\widehat{\mathcal{S}}, \|\cdot\|)}$. It is immediate that the natural basis $(e_t)_{t \in \widehat{\mathcal{S}}}$ is 1-unconditional.

We may enumerate the basis of X as $(e_n)_{n \in \mathbb{N}}$ as follows. Let $\phi : \widehat{\mathcal{S}} \rightarrow \mathbb{N}$ be an 1-1 and onto map such that $\phi(\emptyset) = 1$ and for every $t_1, t_2 \in \widehat{\mathcal{S}}$ with $\max t_1 < \max t_2$, $\phi(t_1) < \phi(t_2)$. We set $e_n = e_{\phi^{-1}(n)}$, for all $n \in \mathbb{N}$. It is easy to see that the space X does not have the property \mathcal{P} (see Definition 6.13). Indeed for every $k \in \mathbb{N}$ let $t_j = \{i : k \leq i < k+j\}$, for all $1 \leq j \leq k$. Then $\|e_{t_j}\| = 1$, for all $1 \leq j \leq k$, and $\|\sum_{j=1}^k e_{t_j}\| = 1$.

Proposition 10.1. The space X admits the usual basis of ℓ^1 as an ω -order spreading model.

PROOF. For every $s \in \mathcal{S}$ let

$$x_s = \sum_{\emptyset \sqsubset t \sqsubseteq s} e_t$$

By the definition of the norm it is easy to see that $\|x_s\| = 1$, for all $s \in \mathcal{S}$, and that $(x_s)_{s \in \mathcal{S}}$ generates the usual basis of ℓ^1 as an \mathcal{S} -spreading model. \square

Our next aim is to show the reflexivity of the space X . To this end we will first show that the space X is ℓ^2 -saturated, that is every subspace of X contains an isomorphic copy of ℓ^2 .

Lemma 10.2. Let $(x_n)_{n \in \mathbb{N}}$ be a seminormalized sequence in X such that for every $n \neq m$ in \mathbb{N} the following are satisfied:

- (i) For every $t_1 \in \text{supp} x_n$ and $t_2 \in \text{supp} x_m$, t_1, t_2 are incomparable.
- (ii) For every $t_1 \in \text{supp} x_n$ and $t_2 \in \text{supp} x_m$, neither the pair (t_1, t_2) nor (t_2, t_1) belongs to \mathcal{H} .

Then the sequence $(x_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^2 .

PROOF. By the definition of the norm we have that if (i) (resp. (ii)) holds then $(x_n)_{n \in \mathbb{N}}$ admits a lower (resp. upper) ℓ^2 estimate. \square

Notation 10.3. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is called $\widehat{\mathcal{S}}$ -block if for every $n < m$ in \mathbb{N} , $t_1 \in \text{supp} x_n$ and $t_2 \in \text{supp} x_m$ we have that $t_1, t_2 \neq \emptyset$ and $\max t_1 < \min t_2$.

The following is immediate from the previous lemma.

Corollary 10.4. Every seminormalized $\widehat{\mathcal{S}}$ -block sequence in X is equivalent to the usual basis of ℓ^2 .

We recall that for every $t \in \widehat{\mathcal{S}}$,

$$\widehat{\mathcal{S}}_{[t]} = \{t' \in \widehat{\mathcal{S}} : t \sqsubseteq t'\}$$

Proposition 10.5. For every $n \geq 2$ the subspace $\overline{c_{00}(\widehat{\mathcal{S}}_{[\{n\}]})}$ is isomorphic to ℓ^2 . More precisely, for every $n \geq 2$ the basis $(e_t)_{t \in \widehat{\mathcal{S}}_{[\{n\}]}}$ is equivalent to the usual basis of ℓ^2 .

PROOF. We will prove it by induction on $n \geq 2$. For $n = 2$ we have the following. Let $k \in \mathbb{N}$ and $a_0, \dots, a_k \in \mathbb{R}$. Then

$$\left\| a_0 e_{\{2\}} + \sum_{j=1}^k a_j e_{\{2, 2+j\}} \right\| = \max \left(|a_0|, \left(\sum_{j=1}^k a_j^2 \right)^{\frac{1}{2}} \right) \leq \left(\sum_{j=0}^k a_j^2 \right)^{\frac{1}{2}}$$

Notice also that $\max(|a_0|^2, \sum_{j=1}^k a_j^2) \geq \frac{1}{2} \sum_{j=0}^k a_j^2$. Hence

$$\frac{1}{\sqrt{2}} \left(\sum_{j=0}^k a_j^2 \right)^{\frac{1}{2}} \leq \left\| a_0 e_{\{2\}} + \sum_{j=1}^k a_j e_{\{2, 2+j\}} \right\| \leq \left(\sum_{j=0}^k a_j^2 \right)^{\frac{1}{2}}$$

and the proof for $n = 2$ is complete. Let $n \geq 2$ and suppose that for every $l \in \mathbb{N}$, $a_1, \dots, a_l \in \mathbb{R}$ and $t_1, \dots, t_l \in \widehat{\mathcal{S}}_{[\{n\}]}$ we have that

$$(\sqrt{2})^{-(n-1)} \left(\sum_{j=1}^l a_j^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{j=1}^l a_j e_{t_j} \right\| \leq \left(\sum_{j=1}^l a_j^2 \right)^{\frac{1}{2}}$$

It is easy to see that for every $k \in \mathbb{N}$ we have that the two 1-unconditional sequences $(e_t)_{t \in \widehat{\mathcal{S}}_{[\{n\}]}}$ and $(e_t)_{t \in \widehat{\mathcal{S}}_{[\{n+1, n+1+k\}]}}$ are 1-equivalent. Hence for every $k \in \mathbb{N}$, $l \in \mathbb{N}$,

$a_1, \dots, a_l \in \mathbb{R}$ and $t_1, \dots, t_l \in \widehat{\mathcal{S}}_{[\{n+1, n+1+k\}]}$ we have that

$$(\sqrt{2})^{-(n-1)} \left(\sum_{j=1}^l a_j^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{j=1}^l a_j e_{t_j} \right\| \leq \left(\sum_{j=1}^l a_j^2 \right)^{\frac{1}{2}}$$

Let $k \in \mathbb{N}$. Let $a_0 \in \mathbb{R}$ and for each $1 \leq j \leq k$, let $l_j \in \mathbb{N}$, $(a_q^j)_{q=1}^{l_j} \in \mathbb{R}$ and $(t_q^j)_{q=1}^{l_j}$ in $\widehat{\mathcal{S}}_{[\{n+1, n+1+j\}]}$. Then we have that

$$\left\| a_0 e_{\{n+1\}} + \sum_{j=1}^k \sum_{q=1}^{l_j} a_q^j e_{t_q^j} \right\| = \max \left\{ |a_0|, \left\| \sum_{j=1}^k \sum_{q=1}^{l_j} a_q^j e_{t_q^j} \right\| \right\} \leq \left(a_0^2 + \sum_{j=1}^k \sum_{q=1}^{l_j} (a_q^j)^2 \right)^{\frac{1}{2}}$$

Notice that $\max \left(a_0^2, \sum_{j=1}^k \sum_{q=1}^{l_j} (a_q^j)^2 \right) \geq \frac{1}{2} \left(a_0^2 + \sum_{j=1}^k \sum_{q=1}^{l_j} (a_q^j)^2 \right)$. Hence

$$(\sqrt{2})^{-n} \left(a_0^2 + \sum_{j=1}^k \sum_{q=1}^{l_j} (a_q^j)^2 \right)^{\frac{1}{2}} \leq \left\| a_0 e_{\{n+1\}} + \sum_{j=1}^k \sum_{q=1}^{l_j} a_q^j e_{t_q^j} \right\| \leq \left(a_0^2 + \sum_{j=1}^k \sum_{q=1}^{l_j} (a_q^j)^2 \right)^{\frac{1}{2}}$$

□

Notation 10.6. For every $l \in \mathbb{N}$ we set $X_l = \overline{c_{00}(\widehat{\mathcal{S}}_{[\{l\}]})}$ and $P_l : X \rightarrow X_l$ such that $P_l(x) = \sum_{t \in \widehat{\mathcal{S}}_{[\{l\}]}} x(t) e_t$, for all $x \in X$. We also set $X_0 = \langle e_\emptyset \rangle$ and $P_0 : X \rightarrow X_0$ such that $P_0(x) = x(\emptyset) e_\emptyset$, for all $x \in X$. Clearly the spaces X_0 and X_1 are of dimension 1 and therefore the projections P_0 and P_1 are compact.

Proposition 10.7. The space X is ℓ^2 saturated.

PROOF. Let Y be a subspace of X . Then either there exists $l \geq 2$ such that $P_l|_Y$ is not strictly singular or for every $l \geq 0$ the operator $P_l|_Y$ is strictly singular.

In the first case the result is immediate. Suppose that the second case holds. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a decreasing sequence of positive reals such that $\sum_{n=1}^{\infty} \varepsilon_n < \frac{1}{3}$. Since P_0 is strictly singular, there exists $y_1 \in S_Y$ such that $\|P_0(y_1)\| < \frac{\varepsilon_1}{2}$. We pick $w_1 \in X$ of finite support such that $\text{supp } w_1 \subseteq \text{supp}(y_1 - P_0(y_1))$ and $\|w_1 - (y_1 - P_0(y_1))\| < \frac{\varepsilon_1}{2}$. Therefore $\|y_1 - w_1\| < \varepsilon_1$ and $\emptyset \notin \text{supp } w_1$. Let $l_1 = \max\{\max t : t \in \text{supp } w_1\}$. Since P_l is strictly singular for all $0 \leq l \leq l_1$, there exists a subspace Y_1 of Y such that $\|P_l|_{Y_1}\| < \frac{\varepsilon_2}{2(l_1+1)}$. Let $y_2 \in S_{Y_1}$ and $\tilde{w}_2 = y_2 - \sum_{l=0}^{l_1} P_l(y_2)$. Then $\|\tilde{w}_2 - y_2\| < \frac{\varepsilon_2}{2}$. Let $w_2 \in X$ of finite support such that $\text{supp } w_2 \subseteq \text{supp } \tilde{w}_2$ and $\|w_2 - \tilde{w}_2\| < \frac{\varepsilon_2}{2}$. Hence $\|w_2 - y_2\| < \varepsilon_2$ and

$$\max\{\max t : t \in \text{supp } w_1\} < \min\{\min t : t \in \text{supp } w_2\}$$

Proceeding in the same way we may construct a normalized sequence $(y_n)_{n \in \mathbb{N}}$ in Y and an $\widehat{\mathcal{S}}$ -block sequence $(w_n)_{n \in \mathbb{N}}$ such that $\|w_n - y_n\| < \varepsilon_n$, for all $n \in \mathbb{N}$. The latter yields that the sequences $(y_n)_{n \in \mathbb{N}}$ and $(w_n)_{n \in \mathbb{N}}$ are equivalent. By Corollary 10.4 we have that $(w_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^2 . Hence $(y_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of ℓ^2 and the proof is complete. □

Proposition 10.8. The space X is reflexive.

PROOF. First recall that the space X has an unconditional basis. Since, by Proposition 10.7, X is ℓ^2 saturated, we have that X does not contain any isomorphic copy of c_0 or ℓ^1 . Hence by James' theorem (c.f. [16]) we have that X is reflexive. □

2. The space X does not admit any plegma block generated ℓ^1 spreading model

Notation 10.9. Let $\mathcal{G} \subseteq [\mathbb{N}]^{<\infty}$, $k \geq 2$ and $s_0, \dots, s_k \in \mathcal{G}$. We will say that $(s_j)_{j=0}^k$ is a 3-plegma path from s_0 to s_k in \mathcal{G} of length k , if (s_j, s_{j+1}, s_{j+2}) is plegma for all $0 \leq j \leq k-2$.

We will need a strengthening of Proposition 1.17.

Proposition 10.10. Let \mathcal{G} be a regular thin family and $L \in [\mathbb{N}]^\infty$ such that \mathcal{G} is very large in L . Then for every $s_0, s \in \mathcal{G} \restriction L(2\mathbb{N})$, with $s_0 < s$, there exists a 3-plegma path from s_0 to s in $\mathcal{G} \restriction L$ of length $2|s_0|$.

PROOF. By Proposition 1.17 there exists a plegma path $(s'_j)_{j=0}^{|s_0|}$ from s_0 to s in $\mathcal{G} \restriction L(2\mathbb{N})$ of length $|s_0|$. For every $1 \leq j \leq |s_0|$, if $s'_j = \{L(n_1^j) < \dots < L(n_{|s'_j|}^j)\}$, we set \tilde{s}_j the unique element of \mathcal{G} such that

$$\tilde{s}_j \sqsubseteq \{L(n_1^j - 1), \dots, L(n_{|s'_j|}^j - 1)\}$$

We set $s_{2j} = s'_j$, for all $0 \leq j \leq |s_0|$, and $s_{2j-1} = \tilde{s}_j$, for all $1 \leq j \leq |s_0|$. It is easy to check that $(s_j)_{j=0}^{2|s_0|}$ is a 3-plegma path from s_0 to s in $\mathcal{G} \restriction L$ of length $2|s_0|$. \square

Let $W \subseteq c_{00}(\widehat{\mathcal{S}})$ be the minimal set satisfying the following.

- (i) For every $l \in \mathbb{N}$, $(t_j)_{j=1}^l \in \mathcal{H}$ and $\varepsilon_1, \dots, \varepsilon_l \in \{-1, 1\}$ the functional $f = \sum_{j=1}^l \varepsilon_j e_{t_j}^*$ belongs to W and will be called of type I with weight $|t_1|$.
- (ii) For every $d \in \mathbb{N}$, every collection f_1, \dots, f_d of functionals of type I, such that the set $\cup_{q=1}^d \text{supp}(f_q)$ is consisted by pairwise \sqsubseteq -incomparable elements, and $a_1, \dots, a_d \in \mathbb{R}$ with $\sum_{q=1}^d a_q^2 \leq 1$ the functional $\varphi = \sum_{q=1}^d a_q f_q$ belongs to W and will be called of type II.

It is easy to check that the set W is a norming set for the space X . Moreover, for $d \in \mathbb{N}$, the collection f_1, \dots, f_d of functionals of type I is called incomparable if the elements of the set $\cup_{q=1}^d \text{supp}(f_q)$ are pairwise \sqsubseteq -incomparable.

Lemma 10.11. Let \mathcal{G} be a regular thin family and $(x_v)_{v \in \mathcal{G}}$ a \mathcal{G} -sequence in B_X . Suppose that there exists $k_0 \in \mathbb{N}$ and $L \in [\mathbb{N}]^\infty$ such that for all $v \in \mathcal{G} \restriction L$ and $t \in \text{supp}(x_v)$, $|t| \leq k_0$ and for every plegma pair (v_1, v_2) in $\mathcal{G} \restriction L$, we have that $\text{supp}(x_{v_1}) \cap \text{supp}(x_{v_2}) = \emptyset$. Then the \mathcal{G} -subsequence $(x_v)_{v \in \mathcal{G} \restriction L}$ does not admit ℓ^1 as a \mathcal{G} -spreading model.

PROOF. Let $L' \in [L]^\infty$. For every $\varepsilon > 0$ there exists $l \in \mathbb{N}$ such that $\frac{\sqrt{k_0+1}}{l} < \varepsilon$. We pick a plegma l -tuple $(v_j)_{j=1}^l$ in $\mathcal{G} \restriction L'$ with $v_1(1) \geq L'(l)$. We will show that $\|\frac{1}{l} \sum_{j=1}^l x_{v_j}\| < \varepsilon$. Indeed, for every $\varphi \in W$ there exist $d \in \mathbb{N}$, f_1, \dots, f_d of type I incomparable and $a_1, \dots, a_d \in \mathbb{R}$ with $\sum_{q=1}^d a_q^2 \leq 1$, such that $\varphi = \sum_{q=1}^d a_q f_q$. We are interested to estimate the quantity $\varphi(\frac{1}{l} \sum_{j=1}^l x_{v_j})$. Since for every $t \in \text{supp}(\sum_{j=1}^l x_{v_j})$, $|t| \leq k_0$, we may assume that the weight of f_q is at most k_0 , for all $1 \leq q \leq d$. For every $1 \leq q \leq d$, we set

$$E_q = \{j \in \{1, \dots, l\} : \text{supp}(f_q) \cap \text{supp}(x_{v_j}) \neq \emptyset\}$$

Notice that $|E_q| \leq |\text{supp}(f_q)| \leq k_0 + 1$, for all $1 \leq q \leq d$. We have the following

$$\begin{aligned} \left| \varphi \left(\frac{1}{l} \sum_{j=1}^l x_{v_j} \right) \right| &\leq \sum_{q=1}^d \left| \frac{1}{l} a_q f_q \left(\sum_{j=1}^l x_{v_j} \right) \right| \leq \sum_{q=1}^d \sum_{j \in E_q} \left| \frac{1}{l} a_q f_q(x_{v_j}) \right| \\ &\leq \left(\sum_{q=1}^d \sum_{j \in E_q} a_q^2 \right)^{\frac{1}{2}} \left(\sum_{q=1}^d \sum_{j \in E_q} \left(\frac{1}{l} f_q(x_{v_j}) \right)^2 \right)^{\frac{1}{2}} \\ &\leq \sqrt{k_0 + 1} \left(\frac{1}{l^2} \sum_{j=1}^l \sum_{q=1}^d (f_q(x_{v_j}))^2 \right)^{\frac{1}{2}} \leq \frac{\sqrt{k_0 + 1}}{l} < \varepsilon \end{aligned}$$

□

Remark 10.12. Let $x_1, x_2 \in X$ such that $x_1 < x_2$ with respect to $(e_n)_{n \in \mathbb{N}}$, where $e_n = e_{\phi^{-1}(n)}$ for all $n \in \mathbb{N}$. Let $F \subseteq \text{supp}(x_2)$ such that for every $t \in F$ there exists $t' \in \text{supp}(x_2)$ with $t' \sqsubset t$. Then there is no $t_1 \in \text{supp}(x_1)$ and $t_2 \in F$ such that (t_1, t_2) or (t_2, t_1) belong to \mathcal{H} .

Indeed, let $t_1 \in \text{supp}(x_1)$ and $t_2 \in F$ with $|t_1| = |t_2|$. The pair $(t_2, t_1) \notin \mathcal{H}$. Indeed, since otherwise we would have that $\max t_2 < \max t_1$ and therefore $\phi(t_2) < \phi(t_1)$, which contradicts that $x_1 < x_2$.

Suppose that the pair $(t_1, t_2) \in \mathcal{H}$. Thus (t_1, t_2) is plegma. Since $t_2 \in F$ there exists $t'_2 \in \text{supp}(x_2)$ such that $t'_2 \sqsubset t_2$. Hence (t_1, t'_2) is plegma and $|t_1| > |t'_2|$. The latter easily yields that $\max t_1 > \max t'_2$. Hence $\phi(t_1) > \phi(t'_2)$, which contradicts that $x_1 < x_2$.

Corollary 10.13. Let \mathcal{G} be a regular thin family, $l \in \mathbb{N}$ and $(x_v)_{v \in \mathcal{G}}$ a \mathcal{G} -sequence in B_X such that for every plegma pair (v_1, v_2) in $\mathcal{G} \upharpoonright L$, $x_{v_1} < x_{v_2}$. Suppose that for every $v \in \mathcal{G} \upharpoonright L$ there exist $F_v^1, F_v^2 \subseteq \text{supp}(x_v)$ such that for every $t \in F_v^2$ there exists $t' \in F_v^1$ such that $t' \sqsubset t$. Let $x_v^2 = x_v|_{F_v^2}$ for every $v \in \mathcal{G} \upharpoonright L$. Then $(x_v^2)_{v \in \mathcal{G} \upharpoonright L}$ does not admit ℓ^1 as a \mathcal{G} -spreading model.

PROOF. Indeed, let $L' \in [L]^\infty$. For every $\varepsilon > 0$ we pick $l_0 \in \mathbb{N}$ such that $\frac{1}{l_0} < \varepsilon$. Let $(v_j)_{j=1}^{l_0}$ be a plegma l_0 -tuple in $\mathcal{G} \upharpoonright L'$. We will show that $\|\frac{1}{l_0} \sum_{j=1}^{l_0} x_{v_j}^2\| < \varepsilon$. Indeed let $\varphi \in W$. Then there exist $d \in \mathbb{N}$, f_1, \dots, f_d of type I incomparable and $a_1, \dots, a_d \in \mathbb{R}$ with $\sum_{q=1}^d a_q^2 \leq 1$ such that $\varphi = \sum_{q=1}^d a_q f_q$. For every $1 \leq q \leq d$, we set $E_q = \{j \in \{1, \dots, l_0\} : \text{supp}(f_q) \cap \text{supp}(x_{v_j}^2) \neq \emptyset\}$. By Remark 10.12 we have that $|E_q| \leq 1$ for all $1 \leq q \leq d$. Hence

$$\begin{aligned} \left| \varphi \left(\frac{1}{l_0} \sum_{j=1}^{l_0} x_{v_j}^2 \right) \right| &\leq \sum_{q=1}^d \left| \frac{1}{l_0} a_q f_q \left(\sum_{j=1}^{l_0} x_{v_j}^2 \right) \right| \leq \sum_{q=1}^d \sum_{j \in E_q} \left| \frac{1}{l_0} a_q f_q(x_{v_j}^2) \right| \\ &\leq \left(\sum_{q=1}^d \sum_{j \in E_q} a_q^2 \right)^{\frac{1}{2}} \left(\sum_{q=1}^d \sum_{j \in E_q} \left(\frac{1}{l_0} f_q(x_{v_j}^2) \right)^2 \right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{l_0^2} \sum_{j=1}^{l_0} \sum_{q=1}^d (f_q(x_{v_j}^2))^2 \right)^{\frac{1}{2}} \leq \frac{1}{l_0} < \varepsilon \end{aligned}$$

□

Lemma 10.14. Let \mathcal{G} be a regular thin family, $L \in [\mathbb{N}]^\infty$ and $(x_v)_{v \in \mathcal{G}}$ a \mathcal{G} -sequence in B_X . Let $k_0 \in \mathbb{N}$ and set for every $v \in \mathcal{G} \upharpoonright L$,

$$F_v^1 = \{t \in \text{supp}(x_v) : |t| \leq k_0\} \text{ and} \\ F_v^3 = \{t \in \text{supp}(x_v) : \forall t' \in F_v^1, \ t, t' \text{ are incomparable}\}$$

Suppose that there exists $\delta < 1$ such that for every $v \in \mathcal{G} \upharpoonright L$, $\|x_v|_{F_v^3}\| \leq \delta$. Then $(x_v)_{v \in \mathcal{G} \upharpoonright L}$ does not admit the usual basis of ℓ^1 as a plegma block generated \mathcal{G} -spreading model.

PROOF. For every $v \in \mathcal{G} \upharpoonright L$ we define

$$F_v^2 = \{t \in \text{supp}(x_v) \setminus F_v^1 : \exists t' \in F_v^1 \text{ such that } t' \sqsubset t\}$$

It is immediate that for every $v \in \mathcal{G} \upharpoonright L$, $(F_v^i)_{i=1}^3$ is a partition of $\text{supp}(x_v)$ and for every $t \in F_v^2 \cup F_v^3$ we have that $|t| > k_0$. For every $v \in \mathcal{G} \upharpoonright L$ and $1 \leq i \leq 3$ we set $x_v^i = x_v|_{F_v^i}$.

Suppose on the contrary that there exists $L_1 \in [L]^\infty$ such that the \mathcal{G} -subsequence $(x_v)_{v \in \mathcal{G} \upharpoonright L_1}$ plegma block generates the usual basis of ℓ^1 as a \mathcal{G} -spreading model. By Lemma 10.11 (resp. Corollary 10.13) we have that $(x_v^1)_{v \in \mathcal{G} \upharpoonright L_1}$ (resp. $(x_v^2)_{v \in \mathcal{G} \upharpoonright L_1}$) does not admit ℓ^1 as a \mathcal{G} -spreading model. Hence by Corollary 4.2 $(x_v^3)_{v \in \mathcal{G} \upharpoonright L_1}$ admits the usual basis of ℓ^1 as a \mathcal{G} -spreading model, which is impossible since $\|x_v^3\| \leq \delta < 1$, for all $v \in \mathcal{G} \upharpoonright L_1$. \square

Notation 10.15. Let

$$D = \left\{ (\varepsilon, \delta) \in [0, \frac{1}{3} - \frac{\sqrt{3}}{6}) \times [0, 1) : (\sqrt{3} - 3\sqrt{3}\varepsilon)^2 + (1 - 3\varepsilon - \delta)^2 \geq 3 \right\}$$

and $h : D \rightarrow \mathbb{R}$ be the function defined by

$$h(\varepsilon, \delta) = (1 - (1 - 3\varepsilon - (3 - 2(1 - 3\varepsilon)^2 - (1 - 3\varepsilon - \delta)^2)^{\frac{1}{2}})^2)^{\frac{1}{2}}$$

Let us note that the curve $\mathcal{E} = \{(\varepsilon, \delta) \in \mathbb{R}^2 : (\sqrt{3} - 3\sqrt{3}\varepsilon)^2 + (1 - 3\varepsilon - \delta)^2 = 3\}$ is an ellipse, since its image through the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined by

$$T(\varepsilon, \delta) = \begin{bmatrix} 3\sqrt{3} & 0 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}$$

is a circle centered at $(\sqrt{3}, 1)$ and of radius $\sqrt{3}$. Moreover notice that $(\frac{1}{3} - \frac{\sqrt{3}}{6}, 0)$ is the first intersection point of the curve \mathcal{E} and the ε -axis. Also the point $(0, 1)$ belongs to \mathcal{E} and the δ -axis is the tangent of \mathcal{E} at $(0, 1)$. Therefore the set D is a curved triangle with edges J_1, J_2, J_3 , where J_1 (respectively J_2) is the segment with endpoints $(\frac{1}{3} - \frac{\sqrt{3}}{6}, 0)$ and $(0, 0)$ (respectively $(0, 0)$ and $(0, 1)$) and J_3 is the arc of \mathcal{E} which joins the $(0, 1)$ with $(\frac{1}{3} - \frac{\sqrt{3}}{6}, 0)$.

It is easy to see that the function h is well defined on D . Moreover the function h is strictly increasing on D in the following sense: for all $(\varepsilon', \delta'), (\varepsilon, \delta) \in D$, with either $0 \leq \varepsilon' < \varepsilon$ and $0 \leq \delta' \leq \delta$ or $0 \leq \varepsilon' \leq \varepsilon$ and $0 \leq \delta' < \delta$, we have that $h(\varepsilon', \delta') < h(\varepsilon, \delta)$. Finally $h[D] = [0, 1]$, $h^{-1}(\{0\}) = \{(0, 0)\}$ and $h^{-1}(\{1\}) = J_3$.

Lemma 10.16. Let $x_1 < x_2 < x_3$ (with respect to $(e_n)_{n \in \mathbb{N}}$, where $e_n = e_{\phi^{-1}(\{n\})}$ for all $n \in \mathbb{N}$) in B_X and $k_0 \in \mathbb{N}$. For every $j = 1, 2, 3$ let

$$\begin{aligned} F_j^1 &= \{t \in \text{supp}(x_j) : |t| \leq k_0\} \\ F_j^2 &= \{t \in \text{supp}(x_j) \setminus F_j^1 : \exists t' \in F_j^1 \text{ such that } t' \sqsubset t\} \\ F_j^3 &= \{t \in \text{supp}(x_j) : \forall t' \in F_j^1, \ t, t' \text{ incomparable}\} \\ x_j^1 &= x_j|_{F_j^1}, x_j^2 = x_j|_{F_j^2} \text{ and } x_j^3 = x_j|_{F_j^3} \end{aligned}$$

Let $(\varepsilon, \delta) \in D$. Suppose that the following are satisfied:

- (i) $\|x_1 + x_2 + x_3\| > 3 - 3\varepsilon$ and
- (ii) $\|x_2^3\| < \delta$.

Then $\|x_3^3\| < h(\varepsilon, \delta)$.

PROOF. Since $\|x_1 + x_2 + x_3\| > 3 - 3\varepsilon$, there exists $\varphi \in W$ such that $\varphi(x_1 + x_2 + x_3) > 3 - 3\varepsilon$. Then there exist $d \in \mathbb{N}$, f_1, \dots, f_d of type I incomparable and $a_1, \dots, a_d \in \mathbb{R}$, with $\sum_{q=1}^d a_q^2 \leq 1$ such that $\varphi = \sum_{q=1}^d a_q f_q$. Let $I = \{1, \dots, d\}$. For every $F \subseteq \{1, 2, 3\}$ nonempty we set

$$\begin{aligned} I_F &= \{q \in I : \text{supp}(f_q) \cap \text{supp}(x_i) \neq \emptyset, \ \forall i \in F, \\ &\quad \text{and } \text{supp}(f_q) \cap \text{supp}(x_i) = \emptyset, \ \forall i \notin F\} \end{aligned}$$

and $\varphi_F = \sum_{q \in I_F} a_q f_q$. Moreover we set

$$\begin{aligned} I_{\leq k_0} &= \{q \in I_{\{1,2,3\}} : w(f_q) \leq k_0\}, \quad I_{> k_0} = \{q \in I_{\{1,2,3\}} : w(f_q) > k_0\}, \\ \varphi_{\leq k_0} &= \sum_{q \in I_{\leq k_0}} a_q f_q \text{ and } \varphi_{> k_0} = \sum_{q \in I_{> k_0}} a_q f_q \end{aligned}$$

Since $\varphi(x_1 + x_2 + x_3) > 3 - 3\varepsilon$ and $\varphi(x_i) \leq 1$, for all $1 \leq i \leq 3$, we have that $\varphi(x_i) > 1 - 3\varepsilon$, for all $1 \leq i \leq 3$. Hence

$$\begin{aligned} 1 - 3\varepsilon < \varphi(x_1) &= \sum_{1 \in F \subseteq \{1,2,3\}} \varphi_F(x_1) = \sum_{1 \in F \subseteq \{1,2,3\}} \sum_{q \in I_F} a_q f_q(x_1) \\ &\leq \left(\sum_{1 \in F \subseteq \{1,2,3\}} \sum_{q \in I_F} a_q^2 \right)^{\frac{1}{2}} \end{aligned}$$

Thus

$$\left(\sum_{q \in I_{\{2\}} \cup I_{\{3\}} \cup I_{\{2,3\}}} a_q^2 \right)^{\frac{1}{2}} < (1 - (1 - 3\varepsilon)^2)^{\frac{1}{2}}$$

Similarly by $1 - 3\varepsilon < \varphi(x_3)$ we get that

$$\left(\sum_{q \in I_{\{1\}} \cup I_{\{2\}} \cup I_{\{1,2\}}} a_q^2 \right)^{\frac{1}{2}} < (1 - (1 - 3\varepsilon)^2)^{\frac{1}{2}}$$

By Remark 10.12 we have that $\text{supp}(\varphi_{\{1,2,3\}}) \cap \text{supp}(x_2^2) = \emptyset$. Hence it is easy to see that

$$\varphi_{\{1,2,3\}}(x_2) = \varphi_{\{1,2,3\}}(x_2^1 + x_2^3) = \varphi_{\leq k_0}(x_2^1) + \varphi_{> k_0}(x_2^3)$$

Thus

$$1 - 3\varepsilon < \varphi(x_2) = \sum_{\substack{2 \in F \subseteq \{1,2,3\} \\ F \neq \{1,2,3\}}} \sum_{q \in I_F} a_q f_q(x_2) + \varphi_{\leq k_0}(x_2^1) + \varphi_{> k_0}(x_2^3)$$

Since $\|x_2^3\| < \delta$, we have that

$$1 - 3\varepsilon - \delta < \left(\sum_{\substack{2 \in F \subseteq \{1,2,3\} \\ F \neq \{1,2,3\}}} \sum_{q \in I_F} a_q^2 + \sum_{q \in I_{\leq k_0}} a_q^2 \right)^{\frac{1}{2}}$$

Hence

$$\left(\sum_{q \in I_{\{1\}} \cup I_{\{3\}} \cup I_{\{1,3\}} \cup I_{>k_0}} a_q^2 \right)^{\frac{1}{2}} < (1 - (1 - 3\varepsilon - \delta)^2)^{\frac{1}{2}}$$

By the above we have that

$$\left(\sum_{\substack{F \subseteq \{1,2,3\} \\ F \neq \emptyset, \{1,2,3\}}} \sum_{q \in I_F} a_q^2 + \sum_{q \in I_{>k_0}} a_q^2 \right)^{\frac{1}{2}} < (3 - 2(1 - 3\varepsilon)^2 - (1 - 3\varepsilon - \delta)^2)^{\frac{1}{2}}$$

The latter yields that

$$\sum_{\substack{F \subseteq \{1,2,3\} \\ F \neq \emptyset, \{1,2,3\}}} \varphi_F(x_3) + \varphi_{>k_0}(x_3) < (3 - 2(1 - 3\varepsilon)^2 - (1 - 3\varepsilon - \delta)^2)^{\frac{1}{2}}$$

Hence $\varphi_{\leq k_0}(x_3) > 1 - 3\varepsilon - (3 - 2(1 - 3\varepsilon)^2 - (1 - 3\varepsilon - \delta)^2)^{\frac{1}{2}}$. Since $\varphi_{\leq k_0}(x_3) = \varphi_{\leq k_0}(x_3^1)$ we have that $\|x_3^1\| > 1 - 3\varepsilon - (3 - 2(1 - 3\varepsilon)^2 - (1 - 3\varepsilon - \delta)^2)^{\frac{1}{2}}$. Since $1 \geq \|x_3\| \geq \|x_3^1 + x_3^3\| \geq (\|x_3^1\|^2 + \|x_3^3\|^2)^{\frac{1}{2}}$, we have that

$$\|x_3^3\| < (1 - (1 - 3\varepsilon - (3 - 2(1 - 3\varepsilon)^2 - (1 - 3\varepsilon - \delta)^2)^{\frac{1}{2}})^2)^{\frac{1}{2}} = h(\varepsilon, \delta)$$

□

The proof of the following lemma is similar to the above and we omit it.

Lemma 10.17. Let $x_1 < x_2 < x_3$ in B_X and $\varepsilon > 0$ such that $(\varepsilon, 0) \in D$ and $\|x_1 + x_2 + x_3\| > 3 - 3\varepsilon$. We set

$$k_0 = \max\{|t| : t \in \text{supp}(x_1)\}, \quad F_3^1 = \{t \in \text{supp}(x_3) : |t| \leq k_0\}, \\ F_3^3 = \{t \in \text{supp}(x_3) : \forall t' \in F_3^1, \quad t, t' \text{ incomparable}\} \quad \text{and} \quad x_3^3 = x_3|_{F_3^3}$$

Then $\|x_3^3\| < h(\varepsilon, 0)$.

Theorem 10.18. The space X does not contain any plegma block generated ℓ^1 spreading model.

PROOF. Suppose on the contrary that X admits ℓ^1 as a plegma block generated ξ -order spreading model, for some $\xi < \omega_1$. Then by Corollary 5.17 the space X admits the usual basis of ℓ^1 as a plegma block generated $(\xi + 1)$ -order spreading model. That is there exist a regular thin family of \mathcal{G} order $\xi + 1$, $M \in [\mathbb{N}]^\infty$ and a \mathcal{G} -sequence $(x_v)_{v \in \mathcal{G}}$ such that the \mathcal{G} -subsequence $(x_v)_{v \in \mathcal{G} \upharpoonright M}$ plegma block generates the usual basis of ℓ^1 as a \mathcal{G} -spreading model. Clearly we may suppose that the \mathcal{G} -sequence $(x_v)_{v \in \mathcal{G}}$ is normalized.

We inductively choose sequence $(\delta_n)_{n=0}^\infty$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ as follows. We set $\delta_0 = 0$ and we pick $0 < \varepsilon_1 < \frac{1}{3} - \frac{\sqrt{3}}{6}$. Then $(\varepsilon_1, \delta_0) \in D \setminus J_3$ and therefore $0 < h(\varepsilon_1, \delta_0) < 1$. We set $\delta_1 = h(\varepsilon_1, \delta_0)$. Suppose that $\varepsilon_1 > \dots > \varepsilon_n$ and $\delta_0, \dots, \delta_n$ have been chosen such that for every $1 \leq k \leq n$

$$0 < \delta_k = h(\varepsilon_k, \delta_{k-1}) < 1$$

We pick $\varepsilon_{n+1} < \varepsilon_n$, such that $(\varepsilon_{n+1}, \delta_n) \in D \setminus J_3$. Thus $0 < h(\varepsilon_{n+1}, \delta_n) < 1$ and we set

$$\delta_{n+1} = h(\varepsilon_{n+1}, \delta_n)$$

It is clear that for every $n \in \mathbb{N}$, $\varepsilon_n > \varepsilon_{n+1}$ and $0 < \delta_n < 1$.

We pass to $M_1 \in [M]^\infty$ such that \mathcal{G} is very large in M_1 and the \mathcal{G} -subsequence $(x_v)_{v \in \mathcal{G} \upharpoonright M_1}$ plegma block generates the usual basis of ℓ^1 as a \mathcal{G} -spreading model with respect to $(\frac{\varepsilon_n}{2})_{n \in \mathbb{N}}$. Let v_0 be the unique element of \mathcal{G} such that $v_0 \sqsubset M_1(4\mathbb{N})$ and $k_0 = \max\{|t| : t \in \text{supp}(x_{v_0})\}$. For every $v \in \mathcal{G}$ we set

$$\begin{aligned} F_v^1 &= \{t \in \text{supp}(x_v) : |t| \leq k_0\}, \\ F_v^2 &= \{t \in \text{supp}(x_v) \setminus F_v^1 : \exists t' \in F_v^1 \text{ such that } t' \sqsubset t\}, \\ F_v^3 &= \{t \in \text{supp}(x_v) : \forall t' \in F_v^1, t', t \text{ are incomparable}\}, \\ x_v^1 &= x_v|_{F_v^1}, \quad x_v^2 = x_v|_{F_v^2} \text{ and } x_v^3 = x_v|_{F_v^3} \end{aligned}$$

Let also $M_2 = \{n \in M_1(4\mathbb{N}) : n > \max v_0\}$.

Claim: For every $v \in \mathcal{G} \upharpoonright M_2$ we have that $\|x_v^3\| < \delta_{2|v_0|}$.

PROOF OF CLAIM. Let $v \in \mathcal{G} \upharpoonright M_2$. By Corollary 10.10 there exists a strong plegma path $(v_j)_{j=0}^{2|v_0|}$ from v_0 to v in $\mathcal{G} \upharpoonright M_1$ of length $2|v_0|$. Since (v_0, v_1, v_2) is plegma we have that $x_{v_0} < x_{v_1} < x_{v_2}$ and $\|x_{v_0} + x_{v_1} + x_{v_2}\| > 3 - 3\varepsilon_1$. By Lemma 10.17 we have that $\|x_{v_1}^3\| < \delta_1$. Inductively for $j = 1, \dots, 2|v_0| - 1$ we have that (v_{j-1}, v_j, v_{j+1}) is plegma. Consequentially we have that $x_{v_{j-1}} < x_{v_j} < x_{v_{j+1}}$. Notice also that $\|x_{v_{j-1}} + x_{v_j} + x_{v_{j+1}}\| > 3 - 3\varepsilon_{j+1}$. Having inductively that $\|x_{v_j}^3\| < \delta_j$ (notice that for $j = 1$ it is true) by Lemma 10.16 we get that $\|x_{v_{j+1}}^3\| < h(\varepsilon_j, \delta_j) = \delta_{j+1}$. Hence $\|x_v^3\| = \|x_{v_{2|v_0|}}^3\| < \delta_{2|v_0|}$. \square

The validity of the above claim and the observation that the \mathcal{G} -subsequence $(x_v)_{v \in \mathcal{G} \upharpoonright M_2}$ also plegma block generates the usual basis of ℓ^1 as a \mathcal{G} -spreading model contradicts Lemma 10.14. \square

By the reflexivity of the space X , Theorem 10.18 and Proposition 7.24 we have the following.

Corollary 10.19. The dual space X^* of X does not admit c_0 as spreading model of any order.

CHAPTER 11

A reflexive space not admitting ℓ^p or c_0 as a spreading model

In this chapter we present an example of a reflexive space X having the property that every spreading model, of any order, of X does not contain any isomorphic copy of c_0 or ℓ^p , for every $p \in [1, \infty)$. This example answers in the affirmative a related problem posed in [24] and shows that Krivine's theorem [19] concerning ℓ^p or c_0 block finite representability cannot be captured by the notion of spreading models.

1. The definition of the space X

We start with the definition of the space. Its construction is closely related to the corresponding one in [24]. Let $(n_j)_{j \in \mathbb{N}}$ and $(m_j)_{j \in \mathbb{N}}$ be two strictly increasing sequences of natural numbers satisfying the following:

- (i) $\sum_{j=1}^{\infty} \frac{1}{m_j} \leq 0, 1$.
- (ii) For every $a > 0$, we have that $\frac{n_j^a}{m_j} \xrightarrow{j \rightarrow \infty} \infty$.
- (iii) For every $j \in \mathbb{N}$, we have that $\frac{n_j}{n_{j+1}} < \frac{1}{m_j}$.

We consider the minimal subset $W \subset (c_{00}(\mathbb{N}))^\#$ satisfying the following:

- (i) $\pm e_n^* \in W$, for all $n \in \mathbb{N}$.
- (ii) Functionals of type I: For every $j \in \mathbb{N}$, $d \leq n_j$ and $f_1 < \dots < f_d$ in W , the functional $\varphi = \frac{1}{m_j} \sum_{q=1}^d f_q$ belongs to W . The functional φ is defined to be of type I and we associate to it its weight to be $w(\varphi) = m_j$.
- (iii) Functionals of type II: For every $d \in \mathbb{N}$, $a_1, \dots, a_d \in \mathbb{R}$ with $\sum_{k=1}^d a_k^2 \leq 1$ and f_1, \dots, f_d in W of type I with pairwise different weights, the functional $\varphi = \sum_{k=1}^d a_k f_k$ belongs to W and is defined to be of type II.

We define the norm $\|\cdot\|$ on $c_{00}(\mathbb{N})$, by setting for every $x \in c_{00}(\mathbb{N})$

$$\|x\| = \sup\{\varphi(x) : \varphi \in W\}$$

Let X be the completion of $c_{00}(\mathbb{N})$ under the above norm. It is easy to see that the Hamel basis $(e_n)_{n \in \mathbb{N}}$ of $c_{00}(\mathbb{N})$ is an unconditional basis of the space X .

Also for every $j \in \mathbb{N}$ we define on X the norm $\|\cdot\|_j$ by setting for every $x \in X$

$$\|x\|_j = \sup\{f(x) : f \text{ is of type I with } w(f) = m_j\}$$

Notice that for every $j \in \mathbb{N}$ the norms $\|\cdot\|$ and $\|\cdot\|_j$ are equivalent. Precisely it is easily shown that for every $x \in X$, $\|x\|_j \leq \|x\| \leq m_j \|x\|_j$. It is also easy to check that for every $x \in X$ we have that

$$\|x\| = \max \left\{ \|x\|_\infty, \left(\sum_{j=1}^{\infty} \|x\|_j^2 \right)^{\frac{1}{2}} \right\}$$

where $\|\cdot\|_\infty$ denotes the supremum norm. Hence for every $x \in X$ the sequence $w = (\|x\|_j)_{j \in \mathbb{N}}$ belongs to ℓ^2 and $(\sum_{j=1}^\infty \|x\|_j^2)^{\frac{1}{2}} = \|w\|_{\ell^2} \leq \|x\|$.

2. On the spreading models of the space X

In this section we will show that every spreading model, of any order, of X does not contain any isomorphic copy of c_0 or ℓ^p , for every $p \in [1, \infty)$.

2.1. The space X does not admit ℓ^1 as a spreading model. We first show that X does not admit ℓ^1 as a spreading model of X of any order. We start with the following lemma.

Lemma 11.1. Let $j > 1$ a natural number and $(x_p)_{p=1}^{m_j}$ a block sequence in B_X . Then for every f in W of type I with $w(f) < m_j$ we have that

$$\left| f\left(\frac{x_1 + \dots + x_{n_j}}{n_j}\right) \right| < \frac{2}{w(f)}$$

PROOF. Let $j > 0$ and f in X of type I such that $w(f) = m_i < m_j$. Then there exist $1 \leq d \leq n_i$ and $f_1 < \dots < f_d$ in W , such that $f = \frac{1}{m_i} \sum_{q=1}^d f_q$. Let $(x_p)_{p=1}^{m_j}$ be a block sequence in B_X . We set $I = \{1, \dots, n_j\}$,

$$A = \{p \in I : \text{there exists at most one } q \in \{1, \dots, n_i\} \text{ such that } \text{supp} f_q \cap \text{supp} x_p \neq \emptyset\} \text{ and}$$

$$B = \{p \in I : \text{there exists at least two } q \in \{1, \dots, n_i\} \text{ such that } \text{supp} f_q \cap \text{supp} x_p \neq \emptyset\}$$

Clearly we have that $A \cup B = \emptyset$ and $A \cap B = \emptyset$. It is also easy to see that $|B| \leq d - 1 < n_i$ and $|f(x_p)| \leq \frac{1}{m_i}$, for all $p \in A$. Hence

$$\begin{aligned} \left| f\left(\frac{1}{n_j} \sum_{p=1}^{n_j} x_p\right) \right| &\leq \frac{1}{n_j} \sum_{p=1}^{n_j} |f(x_p)| = \frac{1}{n_j} \sum_{p \in A} |f(x_p)| + \frac{1}{n_j} \sum_{p \in B} |f(x_p)| \\ &\leq \frac{1}{m_i} + \frac{n_i}{n_j} \leq \frac{1}{m_i} + \frac{n_i}{n_{i+1}} \leq \frac{2}{m_i} = \frac{2}{w(f)} \end{aligned}$$

□

For the next proposition we will use the following notation. Given $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$ and $\mathbf{b} = (b_n)_{n \in \mathbb{N}}$ two sequences in $\ell^2(\mathbb{N})$ we set $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{n=1}^\infty a_n b_n$. Also every $F \subseteq \mathbb{N}$ we set $P_F(\mathbf{a}) = \sum_{n \in F} a_n e_n$.

We will also use the next remark.

Remark 11.2. Let \mathcal{F} be regular thin, $M \in [\mathbb{N}]^\infty$ and $(w_s)_{s \in \mathcal{F} \upharpoonright M}$ be a bounded \mathcal{F} -subsequence in ℓ^2 . Hence, since $(w_s)_{s \in \mathcal{F} \upharpoonright M}$ is weakly relatively compact, by Proposition 4.14, we have that for every $\varepsilon > 0$, $d \in \mathbb{N}$ and a decreasing null sequence (ε_n) of reals, there exists $L \in [M]^\infty$ such that the \mathcal{F} -subsequence $(w_s)_{s \in \mathcal{F} \upharpoonright L}$ admits a generic assignment $(\hat{\varphi}, (\tilde{w}_s)_{s \in \mathcal{F} \upharpoonright L}, (\tilde{v}_t)_{t \in \hat{\mathcal{F}} \upharpoonright L})$ with respect to $d \in \mathbb{N}$ and (ε_n) . Notice that we may assume that $\|P_{\{d+1, \dots\}}(\hat{\varphi}(\emptyset))\| < \varepsilon$ and $\varepsilon_n < \varepsilon$ for all $n \in \mathbb{N}$.

Proposition 11.3. The space X does not admit ℓ^1 as plegma block generated spreading model of any order.

PROOF. Assume on the contrary that there exist \mathcal{F} regular thin, $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ which plegma block generates ℓ^1 as an \mathcal{F} -spreading model. We may also assume that $x_s \in B_X$ for all $s \in \mathcal{F} \upharpoonright M$ and $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ plegma block generates ℓ^1 as an \mathcal{F} -spreading model with constants $1 - \varepsilon$, where $\varepsilon = 0, 1$.

For every $s \in \mathcal{F} \upharpoonright M$ we define $w_s = (\|x_s\|_j)_{j \in \mathbb{N}}$ which clearly belongs to B_{ℓ^2} . By Remark 11.2 there exists $L \in [M]^\infty$ such that the \mathcal{F} -subsequence $(w_s)_{s \in \mathcal{F} \upharpoonright L}$ admits a generic assignment $(\widehat{\varphi}, (\tilde{w}_s)_{s \in \mathcal{F} \upharpoonright L}, (\tilde{v}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright L})$ with respect to some $d_0 \in \mathbb{N}$ and (ε_n) decreasing null sequence of reals such that $\|P_{\{d_0+1, \dots\}}(\widehat{\varphi}(\emptyset))\| < \frac{\varepsilon}{2}$ and $\varepsilon_n < \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}$. We pick $j_0 \in \mathbb{N}$ such that

- (i) $j_0 > d_0$.
- (ii) $\frac{\lfloor \frac{1}{\varepsilon_2} \rfloor}{n_{j_0}} < \varepsilon$.

Let $(s_p)_{p=1}^{n_{j_0}}$ be a plegma n_{j_0} -tuple in $\mathcal{F} \upharpoonright L$ with $s_1(1) \geq L(n_{j_0})$ such that

$$(13) \quad \left\| \frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} x_{s_p} \right\| \geq 1 - 2\varepsilon = 0,8$$

Clearly we have that $\left\| \frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} x_{s_p} \right\|_\infty \leq \frac{1}{n_{j_0}} < \varepsilon$. We will arrive to a contradiction by proving that for every φ in W of type II, we have that

$$\varphi\left(\frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} x_{s_p}\right) \leq 0,2 + 4\varepsilon = 0,6$$

Indeed, let φ in W of type II. Then there exist $d \in \mathbb{N}$, $a_1, \dots, a_d \in \mathbb{R}$ with $\sum_{q=1}^d a_q \leq 1$ and f_1, \dots, f_d in W of type I with pairwise different weights such that $\varphi = \sum_{q=1}^d a_q f_q$. We may assume that $w(f_q) = m_q$, for all $1 \leq q \leq d$. Let $\mathbf{a} \in B_{\ell^2}$ such that $\mathbf{a}(q) = a_q$ for all $1 \leq q \leq d$ and $\mathbf{a}(q) = 0$ for all $q > d$. Let also $F_p = \text{supp}(P_{\{d_0+1, \dots\}}(\tilde{w}_{s_p}))$, for all $1 \leq p \leq n_{j_0}$. Then

$$\begin{aligned} \left| \varphi\left(\frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} x_{s_p}\right) \right| &\leq \left| \sum_{q=1}^{d_0} a_q f_q\left(\frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} x_{s_p}\right) \right| + \left| \sum_{q=d_0+1}^d a_q f_q\left(\frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} x_{s_p}\right) \right| \\ &\leq 0,2 + \frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} \sum_{q=d_0+1}^d |a_q| \cdot \|x_{s_p}\|_q \\ &= 0,2 + \frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} \sum_{q=d_0+1}^d |a_q w_{s_p}(q)| \\ &= 0,2 + \frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} |< \mathbf{a}, P_{\{d_0+1, \dots\}}(w_{s_p}) >| \\ &\leq 0,2 + \varepsilon + \frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} |< \mathbf{a}, P_{\{d_0+1, \dots\}}(\tilde{w}_{s_p} - \tilde{v}_0) >| \end{aligned}$$

$$\begin{aligned}
&= 0, 2 + \varepsilon + \frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} | \langle \mathbf{a}, P_{F_p}(\tilde{w}_{s_p} - \tilde{v}_\emptyset) \rangle | \\
&\leq 0, 2 + 2\varepsilon + \frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} | \langle \mathbf{a}, P_{F_p}(w_{s_p}) \rangle | \\
&\leq 0, 2 + 2\varepsilon + \frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} \left(\sum_{q \in F_p} a_q^2 \right)^{\frac{1}{2}}
\end{aligned}$$

Let $A_1 = \{p \in \{1, \dots, n_{j_0}\} : (\sum_{q \in F_p} a_q^2)^{\frac{1}{2}} \leq \varepsilon\}$ and $A_2 = \{p \in \{1, \dots, n_{j_0}\} : (\sum_{q \in F_p} a_q^2)^{\frac{1}{2}} > \varepsilon\}$. Then since $\sum_{q=1}^{n_{j_0}} a_q^2 \leq 1$ and $(F_p)_{p=1}^{n_{j_0}}$ are pairwise disjoint, we have $|A_2| \leq \lfloor \frac{1}{\varepsilon^2} \rfloor$. Hence

$$\begin{aligned}
\left| \varphi \left(\frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} x_{s_p} \right) \right| &\leq 0, 2 + 2\varepsilon + \frac{1}{n_{j_0}} \sum_{p=1}^{n_{j_0}} \left(\sum_{q \in F_p} a_q^2 \right)^{\frac{1}{2}} \\
&= 0, 2 + 2\varepsilon + \frac{1}{n_{j_0}} \sum_{p \in A_1} \left(\sum_{q \in F_p} a_q^2 \right)^{\frac{1}{2}} + \frac{1}{n_{j_0}} \sum_{p \in A_2} \left(\sum_{q \in F_p} a_q^2 \right)^{\frac{1}{2}} \\
&\leq 0, 2 + 2\varepsilon + \varepsilon + \frac{|A_2|}{n_{j_0}} < 0, 2 + 4\varepsilon
\end{aligned}$$

□

Using $\frac{n_j}{m_j} \xrightarrow{j \rightarrow \infty} \infty$ it is easy to see that the space X satisfies the property \mathcal{P} (see Definition 6.13). This easily implies that the space X does not contain any isomorphic copy of c_0 . Proposition 11.3 implies that the space X does not contain any isomorphic copy of ℓ^1 . Since the basis of X is unconditional we conclude the following.

Corollary 11.4. The space X is reflexive.

Moreover we have the following.

Corollary 11.5. The space X does not admit any ℓ^1 spreading model of any order.

PROOF. Suppose on the contrary that there exist a regular thin family \mathcal{F} , $M \in [\mathbb{N}]^\infty$ and a bounded \mathcal{F} -sequence $(x_s)_{s \in \mathcal{F}}$ such that $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ generates ℓ^1 as an \mathcal{F} -spreading model. By the reflexivity of X and the boundness of $(x_s)_{s \in \mathcal{F}}$, we have that the \mathcal{F} -sequence $(x_s)_{s \in \mathcal{F}}$ is weakly relatively compact. Since X satisfies the property \mathcal{P} , by Theorem 6.15 the space X admits ℓ^1 as a plegma block generated \mathcal{F} -spreading model, which contradicts Proposition 11.3. □

2.2. The space X does not admit ℓ^p , for $1 < p$, or c_0 as a spreading model. We proceed to show that X does not admit any ℓ^p , $p > 1$ or c_0 as a spreading model. First we state some preliminary lemmas.

Lemma 11.6. For every $p > 1$ and for every $\delta, c > 0$ there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ and $(x_j)_{j=1}^{n_k}$ block sequence in X with $\|x_j\| > \delta$ for all $1 \leq j \leq n_k$, we have that

$$\left\| \sum_{j=1}^{n_k} x_j \right\| > cn_k^{\frac{1}{p}}$$

PROOF. Since $\frac{n_k}{m_k} \xrightarrow{k \rightarrow \infty} \infty$, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ we have that $\frac{n_k}{m_k} > \frac{1}{\delta}$. Let $(x_j)_{j=1}^{n_k}$ be a block sequence in X such that $\|x_j\| > \delta$ for all $1 \leq j \leq n_k$. Let $f_1, \dots, f_{n_k} \in W$ such that $f_j(x_j) > \delta$ and $\text{supp}(f_j) \subseteq \text{supp}(x_j)$ for all $1 \leq j \leq n_k$. Then the functional $f = \frac{1}{m_k} \sum_{j=1}^{n_k} f_j$ belongs to W . Hence

$$\left\| \sum_{j=1}^{n_k} x_j \right\| \geq f \left(\sum_{j=1}^{n_k} x_j \right) = \frac{1}{m_k} \sum_{j=1}^{n_k} f_j(x_j) > \frac{n_k}{m_k} \delta > cn_k^{\frac{1}{p}}$$

□

Lemma 11.7. Let $p > 1$ and $\xi < \omega_1$. For every regular thin family \mathcal{F} of order ξ , $M \in [\mathbb{N}]^\infty$, $c, \delta > 0$ and \mathcal{F} -sequence $(\tilde{x}_s)_{s \in \mathcal{F}}$, such that $\|\tilde{x}_s\| > \delta$ for all $s \in \mathcal{F} \upharpoonright M$ and $(\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright M}$ admits a disjoint generic decomposition, there exist $L \in [M]^\infty$ and $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$

$$\left\| \sum_{j=1}^{n_k} \tilde{x}_{s_j} \right\| > cn_k^{\frac{1}{p}}$$

for every plegma n_k -tuple $(s_j)_{j=1}^{n_k}$ in $\mathcal{F} \upharpoonright L$.

PROOF. We will prove the lemma using induction on ξ . For $\xi = 1$ we have that the sequence $(\tilde{x}_{\{m\}})_{m \in M}$ forms a block sequence in X . Hence by Lemma 11.6 the result follows.

Let $\xi < \omega_1$ and assume that for every $\zeta < \xi$ the lemma is true. We will show that it also holds for ξ . Indeed, let $c, \delta > 0$, \mathcal{F} be a regular thin family of order ξ , $M \in [\mathbb{N}]^\infty$ and $(\tilde{x}_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence such that the \mathcal{F} -subsequence $(\tilde{x}_s)_{s \in \mathcal{F} \upharpoonright M}$ admits a disjointly generic decomposition $(\tilde{y}_t)_{t \in \widehat{\mathcal{F}} \upharpoonright M}$ and $\|\tilde{x}_s\| > \delta$, for all $s \in \mathcal{F} \upharpoonright M$. Let $L_1 \in [M]^\infty$ such that \mathcal{F} is very large in L_1 . By Proposition 1.12 there exists $L_2 \in [L_1]^\infty$ such that one of the following holds

- (i) $\|\sum_{t \sqsubseteq s_2/s_1} \tilde{y}_t\| \leq \frac{\delta}{2}$ for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L_2$,
- (ii) $\|\sum_{t \sqsubseteq s_2/s_1} \tilde{y}_t\| > \frac{\delta}{2}$ for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L_2$.

Suppose that (i) occurs. Then by Lemma 11.6 there exist k_0 such that for every $k \geq k_0$ and $(x_j)_{j=1}^{n_k}$ block sequence in X with $\|x_j\| > \frac{\delta}{2}$ for all $1 \leq j \leq n_k$, we have that

$$\left\| \sum_{j=1}^{n_k} x_j \right\| > cn_j^{\frac{1}{p}}$$

Let $k \geq k_0$ and $(s_j)_{j=1}^{n_k}$ be a plegma n_k -tuple in $\mathcal{F} \upharpoonright L_2(2\mathbb{N})$. Let $L_2 = \{l_1^2, l_2^2, \dots\}$ and for every $1 \leq j \leq n_k$, let $s_j = \{l_{2\rho_1}^2, \dots, l_{2\rho_{|s_j|}}^2\}$. For all $1 \leq j \leq n_k$ we set s_j^* to be the unique element in $\mathcal{F} \upharpoonright L_2$ with $s_j^* \sqsubseteq \{l_{2\rho_1-1}^2, \dots, l_{2\rho_{|s_j|-1}}^2\}$ and

$$z_{s_j} = \tilde{x}_{s_j} - \sum_{t \sqsubseteq s_j/s_j^*} \tilde{y}_t$$

Notice that the sequence $(z_{s_j})_{j=1}^{n_k}$ forms a block sequence in X such that $\|z_{s_j}\| > \frac{\delta}{2}$ for all $1 \leq j \leq n_k$. Since $k \geq k_0$ and the basis $(e_n)_{n \in \mathbb{N}}$ is 1-unconditional, we get that

$$\left\| \sum_{j=1}^{n_k} \tilde{x}_{s_j} \right\| \geq \left\| \sum_{j=1}^{n_k} z_{s_j} \right\| > cn_j^{\frac{1}{p}}$$

Suppose that (ii) occurs. Let $\mathcal{G} = \mathcal{F}/_{L_2}$. For every $t \in \mathcal{G}$, we set $z_t = \tilde{x}_s^{(1, \mathcal{G})}$ (see Definition 6.3) where $s \in \mathcal{F} \upharpoonright L_2(2\mathbb{N})$ and $t \sqsubset s$. Since $o(\mathcal{G}) < \xi$, by the inductive hypothesis (for $\zeta = o(\mathcal{G})$, $c, \frac{\delta}{2}$, $(z_t)_{t \in \mathcal{G}}$ and $L_2(2\mathbb{N})$), there exist $L \in [L_2(2\mathbb{N})]^\infty$ and $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$,

$$\left\| \sum_{j=1}^{n_k} z_{s_j} \right\| > cn_j^{\frac{1}{p}},$$

for every plegma n_k -tuple $(t_j)_{j=1}^{n_k}$ in $\mathcal{G} \upharpoonright L$.

Let $(s_j)_{j=1}^{n_k}$ plegma n_k -tuple in $\mathcal{F} \upharpoonright L$. For all $1 \leq j \leq n_k$, let t_j in $\mathcal{G} \upharpoonright L$ such that $t_j \sqsubset s_j$. Then $(t_j)_{j=1}^{n_k}$ is an plegma n_k -tuple in $\mathcal{G} \upharpoonright L$ and therefore

$$\left\| \sum_{j=1}^{n_k} \tilde{x}_{s_j} \right\| \geq \left\| \sum_{j=1}^{n_k} z_{s_j} \right\| > cn_j^{\frac{1}{p}}$$

□

The following is immediate by the above.

Corollary 11.8. Let \mathcal{F} be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s \in \mathcal{F}}$ an \mathcal{F} -sequence in X . If the \mathcal{F} -subsequence $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ admits a disjointly generic decomposition then $(x_s)_{s \in \mathcal{F} \upharpoonright M}$ does not admits any ℓ^p , for $1 < p < \infty$, or c_0 as an \mathcal{F} -spreading model.

Moreover since X is reflexive by Theorem 4.15 we get the following.

Corollary 11.9. The space X does not admit any ℓ^p , for $1 < p < \infty$, or c_0 as spreading model of any order.

2.3. The main result and some consequences. We are now ready to state our main results.

Theorem 11.10. Every spreading model, of any order, of the space X does not contain any isomorphic copy of ℓ^p , for every $p \in [1, \infty)$, or c_0 .

PROOF. Assume that for some $\xi < \omega_1$ the space X admits a spreading model of order ξ containing an isomorphic copy of ℓ^p (or c_0). Since X is reflexive by Proposition 5.14 we have that X admits ℓ^p (resp. c_0) as a spreading model of order $\xi + 1$, which contradicts Corollaries 11.9 and 11.5. □

Moreover by Corollaries 11.9, 11.5 and 5.19 we obtain the following.

Corollary 11.11. For every nontrivial spreading model $(e_n)_{n \in \mathbb{N}}$ admitted by X , we have that the space $E = \langle (e_n)_{n \in \mathbb{N}} \rangle$ is reflexive.

Lemma 11.12. For every $k \in \mathbb{N}$ and every Banach space E such that $X \xrightarrow{k} E$, we have that $X \xrightarrow[\text{bl}]{k} E$.

PROOF. For $k = 1$ the result is easily verified by the reflexivity of the space X and the standard sliding hump argument. Suppose that for some $k \in \mathbb{N}$ the lemma is true. Let E be a Banach space such that $X \xrightarrow{k+1} E$. Then there exists a Banach space E' with a Schauder basis $(e'_n)_{n \in \mathbb{N}}$ such that $X \xrightarrow{k} E'$ and $E' \rightarrow E$. By the inductive hypothesis we have that $X \xrightarrow[\text{bl}]{k} E'$. By Corollary 5.9 we have that $(e'_n)_{n \in \mathbb{N}} \in \mathcal{SM}_k(X)$. Therefore by Corollary 11.11 we have that E' is reflexive.

Hence by the standard sliding hump argument, $E' \xrightarrow{\text{bl}} E$. Thus $X \xrightarrow{\text{bl}}^{k+1} E$. By induction on $k \in \mathbb{N}$ the proof is complete. \square

By the above lemma, Corollary 5.9 and Theorem 11.10 we have the following which answers the related question of [24].

Corollary 11.13. For every $k \in \mathbb{N}$ and Banach space E such that $X \xrightarrow{k} E$, we have that E does not contain any isomorphic copy of ℓ^p , for all $1 \leq p \leq \infty$, or c_0 .

Remark 11.14. It is immediate by the reflexivity of the space X , Corollary 11.5 and Theorem 7.26 that the dual space X^* of X does not admit any c_0 spreading model.

It is worth pointing out that the answer to the Problem 2 in Chapter 2 is unknown for the space X . Namely we do not know if there exists an ordinal ξ such that every spreading model of X is equivalent to a ξ -order spreading model of X . It is also unknown if the spreading models of X include reflexive Banach spaces which are totally incomparable to spaces with a saturated norm.

Bibliography

- [1] F. Albiac, N.J. Kalton, *Topics in Banach Space Theory*, Springer, 2006.
- [2] S. A. Argyros, I. Gasparis, *Unconditional structures of weakly null sequences* Trans. Amer. Math. Soc. 353, (2001), no.5, 2019–2058 .
- [3] S. A. Argyros, G. Godefroy, H. P. Rosenthal, *Descriptive set theory and Banach spaces* Handbook of the geometry of Banach spaces, North-Holland, Amsterdam, Vol. 2, (2003), 1007–1069.
- [4] S.A. Argyros, S. Todorćević, *Ramsey methods in analysis*, Birkhauser Verlag, Basel, 2005.
- [5] A. Brunel, L. Sucheston *On B-convex Banach spaces*, Math. Systems Theory 7 (1974), no. 4, 294–299.
- [6] A. Dvoretzky, *Some results on convex bodies and Banach spaces*, 1961 Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960) pp. 123–160 Jerusalem Academic Press, Jerusalem;
- [7] J. Elton, *Weakly null normalized sequences in Banach spaces*, Doctoral thesis, Yale University (1978).
- [8] H. Furstenberg, Y. Katznelson, *An ergodic Szemerédi theorem for commuting transformations* J. Analyse Math. 34 (1978), 275–291.
- [9] F. Galvin, K. Prikry, *Borel sets and Ramsey’s theorem*, J. Symbolic Logic, 38, (1973), 193–198.
- [10] I. Gasparis, *A dichotomy theorem for subsets of the power set of the natural numbers*, Proc. Amer. Math. Soc., 129, (2001), no. 3, 759–764.
- [11] W.T. Gowers, *A Banach space not containing c_0, l_1 or a reflexive subspace*, Trans. Amer. Math. Soc. 344 (1994), no. 1, 407–420.
- [12] W.T. Gowers, *Hypergraph regularity and the multidimensional Szemerédi theorem*, Ann. of Math. (2) 166 (2007), no. 3, 897–946.
- [13] W.T. Gowers, *Ramsey Methods in Banach Spaces*, in Handbook of the geometry of Banach Spaces, vol. 2, (2003), Elsevier Science B.V., 1072–1097.
- [14] W.T. Gowers, B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. 6 (1993), no. 4, 851–874.
- [15] L. Halbeisen, E. Odell, *On asymptotic models in Banach spaces*, Israel J. Math. 139 (2004), 253–291.
- [16] R.C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. (2) 52, (1950), 518–527.
- [17] R.C. James, *Uniformly non-square Banach spaces*, Ann. of Math. (2) 80, (1964), 542–550.
- [18] A.S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, 1995.
- [19] J. L. Krivine, *Sous-espaces de dimension finie des espaces de Banach réticulés*, Ann. of Math (2) 104 (1976), no. 1, 1–29.
- [20] J. Lopez-Abad, S. Todorćević, *Pre-compact families of finite sets of integers and weakly null sequences in Banach spaces*, Topology Appl. 156 (2009), no. 7, 1396–1411.
- [21] S. Mercourakis, *On Cesàro summable sequences of continuous functions*, Mathematika 42 (1995), no. 1, 87–104.
- [22] C.St.J.A. Nash-Williams, *On well quasi-ordering transfinite sequences*, Proc. Cambr. Phil. Soc., 61, (1965), 33–39.
- [23] E. Odell, *On Schreier unconditional sequences*, Contemp. Math. 144, (1993), 197–201.
- [24] E. Odell, Th. Schlumprecht, *On the richness of the set of p ’s in Krivine’s theorem*, Geometric aspects of functional analysis (Israel, 1992–1994), 177–198, Oper. Theory Adv. Appl., 77, Birkhäuser, Basel, 1995.
- [25] P. Pudlak, V. Rödl, *Partition theorems for systems of finite subsets of integers*, Discrete Math. 39, (1982), no. 1, 67–73.
- [26] F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. 30 (2), (1929), 264–286. (2001)